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OF THE
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The Journal is open to contributions from members as well as subscribers. The editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. V. SESHU AIYAR, B.A., L.T., 37, Venkatachala Chetty Street, Triplicane, Madras.

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THE JOURNAL

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Vol. VII.] FEBRUARY 1915. [No. 1.

PROGRESS REPORT.

The following gentlemen have been elected members at the concessional rate.—

1. *Mr. A. G. Chandrachud*, B. A.,—Student. Fergusson College, Poona City.
2. *Mr. T. P. Krishnaswami*, B. A.,—Mathematics Assistant, C. S. School, Srivaikuntham.

With the commencement of the year Messrs. E. W. Middlemast and P. V. Seshu Aiyar have duly taken their seats on the Committee, and Mr. R. Swaminatha Iyer retires from the Committee. Mr. Middlemast has also taken over charge of the office of the President the Society from Prof. R. N. Apte. The best thanks of the Committee are due to the retiring gentlemen for the zealous and conscientious discharge of their duties.

The Calcutta University Calendar for 1914, *Part I*, has been received at the Library.

POONA,
31st January, 1915.

D. D. KAPADIA,
Hony. Joint Secretary.

So-called Cases of Failure in the Solution of Linear Differential Equations.

BY ERIC H. NEVILLE.

There are two ways in which the solution of a particular linear differential equation may "fail" although the solution of a more general equation obtained by replacing certain constants by parameters is complete. The two ways are distinct, but they may both be illustrated by the equation

$$(D-1)^2 y = e^x, \qquad \dots \qquad \dots \qquad \dots \quad (1)$$

where D as usual stands for d/dx.

For the general equation

$$(D-l)\,(D-m)\,y = e^{nx}$$

the perfectly general solution is

$$y = \frac{1}{(n-l)(n-m)} e^{nx} + Ae^{lx} + Be^{mx},$$

A, B being independent arbitrary constants, but if we attempt to apply this solution to the particular equation (1), we find, in the first place, that the coincidence of n with l and m renders the first term infinite and, in the second place, that the coincidence of m with l leaves us with only one effective constant, $A+B$. The method by which in the commoner text-books the passage from the general solution to that of a particular equation is made in such cases as this is unconvincing.

The fundamental problem is always the same: given a linear differential equation

$$G(y) = g(x,\ a_1,\ a_2,\ \dots\ a_r) \qquad \dots \qquad \dots \quad (2)$$

of any order, $a_1, a_2, \dots a_r$ being constants, $G\,(y)$ involving x in any way and being homogeneous in y and a finite number of its derivatives, and $g(x, a_1, a_2 \dots a_r)$ being any function of $x, a_1, a_2, \dots a_r$, and given a function $F\,(x, p_1, p_2, \dots p_r)$ of x and of r parameters $p_1, p_2, \dots p_r$ which is not infinite irrespective of x for the values $a_1, a_2, \dots a_r$ of the parameters and is such that

$$G\,\{\,F(x,\ p_1,\ p_2,\ \dots\ p_r)\,\} = h(p_1,\ p_2,\ \dots\ p_r)\,f\,(x,\ p_1,\ p_2,\ \dots\ p_r), \ \dots\ (3)$$

where the right hand side is not identically zero for all values of $p_1, p_2, \dots p_r$ and for the values $a_1, a_2, \dots a_r$ of $p_1, p_2, \dots p_r$ the function $f(x, p_1, p_2, \dots p_r)$ takes the form $g(x, a_1, a_2, \dots a_r)$; to construct as general a solution as possible of the equation (2).

Thus, in the original example, we know that

$$(D-1)^2 e^{px} = (p-1)^2 e^{px}, \qquad \dots \qquad \dots \quad (4)$$

and we wish to utilise this knowledge in the formation of integrals of the particular equation (1).

It will be noticed that here the function $f(x, p)$, which is e^{px}, is actually the function $g(x, p)$; in practice this is usually the case, but it is quite irrelevant to the argument which follows, which enables us, for example, to deduce a solution of equation (1) from the more elaborate equation

$$(D-1)^2(xe^{px})=2(p-1)\{e^{px}+\tfrac{1}{2}(p-1)xe^{px}\}, \quad \dots \quad (5)$$

taking for $f(x, p)$ the expression $\{e^{px}+\tfrac{1}{2}(p-1)xe^{px}\}$.

We consider first the construction of a particular integral of equation (2), equation (3) being assumed.

If $h(a_1, a_2,\dots a_r)\neq 0$, we have from (3) and the known relation between the functions $f(x, p_1, p_2,\dots p_r)$, $g(x, a_1, a_2,\dots a_r)$, and because the left hand side is linear and homogeneous in y and its derivatives,

$$G\{F(x, a_1, a_2,\dots a_r)/h(a_1, a_2,\dots a_r)\}=g(x, a_1, a_2,\dots a_r),$$

and $$y=F(x, a_1, a_2\dots a_r)/h(a_1, a_2,\dots a_r)$$

is a solution of (2). In any case, from (3), on account of the character of the operator G, we have, p_k being any one of the parameters $p_1, p_2,\dots p_r$,

$$G\{\partial F/\partial p_k\}=(\partial h/\partial p_k)f(x, p_1, p_2\dots p_r)+h(p_1, p_3,\dots p_r)(\partial f/\partial p_k), \quad (6)$$

and precisely, when $h(a_1, a_2,\dots a_r)$ is zero, we can derive from (6) the r equations

$$G\{\partial F/\partial a_k\}=(\partial h/\partial a_k)\, g(x, a_1, a_2,\dots a_r),$$

for $k=1, 2,\dots r$.

If then any of the functions $\partial h/\partial a_1$, $\partial h/\partial a_2,\dots\partial h/\partial a_r$ are not zero, solutions of the original equation (2) are given by

$$y=\frac{\partial F}{\partial a_k}\Big/\frac{\partial h}{\partial a_k}$$

for the corresponding values of k.

Thus from equation (5) we have

$$(D-1)^2\{\partial(xe^{px})/\partial p\}=2\{e^{px}+\tfrac{1}{2}(p-1)xe^{px}\}$$
$$+2(p-1)[\partial\{e^{px}+\tfrac{1}{2}(p-1)xe^{px}\}/\partial p], \quad \dots \quad (7)$$

and from this it follows that

$$y=\tfrac{1}{2}[\partial(xe^{px})/\partial p]_{p=1}1=\tfrac{1}{2}x^2e^x$$

is a solution of equation (1), the right hand side of equation (7) reducing, for $p=1$, to a constant multiple of e^x for the very reason that the factor $p-1$, which prevented our obtaining a solution of equation (1) by mere substitution in equation (5), remains a factor of that part of the expression which is not itself a constant multiple, for $p=1$, of e^x.

But it may happen that all the functions of the form $\partial h/\partial a_k$ are zero. For example, if we are using the simple equation (4) for the purpose of solving equation (1), the function $h(p)$ of the only parameter p is $(p-1)^2$, and dh/dp as well as $h(p)$ is zero when p is equal to 1. And, in fact, differentiation of the equation (4) with respect to p gives us equation (5) itself, in the form

$$(D-1)^2\{\partial e^{px}/\partial p\}=2(p-1)e^{px}+(p-1)^2(\partial e^{px}/\partial p), \quad \dots \quad (8)$$

and the right hand side of this equation does not reduce to a proper multiple of e^x for any value of p.

If however we differentiate the equation (6) with respect to any one of the parameters which it involves, we have for all values of the parameters and for all values of i, k from 1 to r.

$$G\left\{\frac{\partial^2 F}{\partial p_i\,\partial p_k}\right\}=\frac{\partial^2 h}{\partial p_i\,\partial p_k}f+\frac{\partial h}{\partial p_i}\frac{\partial f}{\partial p_k}+\frac{\partial h}{\partial p_k}\frac{\partial f}{\partial p_i}+h\frac{\partial^2 f}{\partial p_i\,\partial p_h}$$

and when h and its first derivatives with respect to the parameters p_i, p_k (which may coincide) vanish for the values $a_1, a_2,\dots a_r$ of $p_1, p_2,\dots p_r$, we have

$$G\{\partial^2 F/\partial a_i\,\partial a_k\}=(\partial^2 h/\partial a_i\partial a_k)g(x, a_1, a_2,\dots a_r)$$

and if the second derivative $\partial^2 h/\partial p_i\,\partial p_k$ of h remains different from zero for the special values of the parameters, we can deduce a solution

$$y=\frac{\partial^2 F}{\partial a_i\,\partial a_k}\bigg/\frac{\partial^2 h}{\partial a_i\,\partial a_k}$$

of our original equation. For example, from equation (8) we have

$$(D-1)^2\{\partial^2 e^{px}/\partial p^2\}=2e^{px}+4(p-1)(\partial e^{px}/\partial p)+(p-1)^2(\partial^2 e^{px}/\partial p^2),$$

an equation which yields us a solution of the equation (1) in the form

$$y=\tfrac{1}{2}(\partial^2 e^{px}/\partial p^2)_{p=1}=\tfrac{1}{2}x^2 e^x$$

simply *because* the earlier equations (4), (8) were ineffective, for this purpose.

The reasoning can be continued from step to step, and we conclude that if $h(p_1, p_2,\dots p_r)$ and all those of its partial derivatives of the first $m-1$ orders which are subsidiary to a particular derivative

$$\partial^m h/\partial p_1{}^{s_1}\partial p^{s_2}\dots\partial p_r{}^{s_r}$$

that is, which are of the form

$$\partial^n h/\partial p_1{}^{t_1}\partial p_2{}^{t_2}\dots\partial p_r{}^{t_r} \text{ where } n<m,\ t_k \overline{<} s_k)$$

vanish when $p_1, p_2,\dots p_r$ have the values $a_1, a_2,\dots a_r$, but this derivative itself does not then vanish, a particular solution of equation (2) is given by

$$y=\frac{\partial^m F}{\partial a_1{}^{s_1}\partial a_2{}^{s_2}\dots\partial a_r{}^{s_r}}\bigg/\frac{\partial^m h}{\partial a_{1s}{}^{1}\partial a_2{}^{s_1}\dots\partial a_r{}^{s_r}}$$

The common case is that in which there is only one parameter p, and only one constant a in the original equation, and the functions $f(x, p)$, $g(x, p)$ are the same; h is a function of p alone, F is a function of x and p, and the result takes the form that if h, dh/dp, d^2h/dp^2, … $d^{m-1}h/dp^{m-1}$ all vanish for $p=a$, but $d^m h/dp^m$ does not, a solution of the equation

$$G(y)=g(x,\ a)$$

is given by

$$y=\frac{\partial^m F}{\partial a^m}\Big/\frac{d^m h}{da^m},$$

$F(x, p)$ and $h(p)$ being such functions that

$$G\{\ F(x,\ p)\ \}=h(p)\cdot g(x,\ p).$$

In this case we obtain only one solution of the equation, but in the general case we often obtain a number of solutions, which may or may not be independent.

To see how a multiplicity of parameters may be utilised, consider the equation

$$(D-1)^2(D-2)y=e^x+e^{2x}, \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

being given that

$$(D-1)^2(D-2)F(x,p,q)=h(p,\ q)\ \{\ e^{px}+e^{qx}\ \}$$

where
$$F(x,\ p,\ q)=(q-1)^2(q-2)e^{px}+(p-1)^2(p-2)e^{qx},$$
$$h(p,\ q)=(p-1)^2(p-2)(q-1)^2(q-2).$$

If we make both $p=1$ and $q=2$ in $h(p, q)$ or in any of its derivatives of the first three orders, except $\partial^3h/\partial p^2\partial q$, the result is zero, but this one derivative has the value -2. Hence one solution of equation (9) is given by

$$y=-\tfrac{1}{2}\frac{\partial^3F(x,\ 1,\ 2)}{\partial p^2\partial q}=-\tfrac{1}{2}x^2e^x+xe^{2x}.$$

It is easy to see that in this example no other integral is derivable from the functions $F(x, p, q)$, $h(p, q)$.

For a second example, in which we cannot, without the use of complex numbers, resolve the problem into two simpler ones, as we should do in practice in the last example, we may solve the equation

$$(D^2-2D+2)y=e^x\sin x, \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

premising that

$$(D^2-2D+2)F(x,\ p,\ q)=h(p,\ q)e^{px}\sin qx \quad \ldots \quad \ldots \quad (11)$$

where

$$F(x,\ p,\ q)=e^{px}[\ \{\ (p-1)^2+(1-q^2)\ \}\sin qx-2(p-1)q\cos qx], \ldots \quad (12)$$
$$h(p,\ q)=\{\ (p-1)^2+(1-q^2)\ \}^2+4(p-1)^2q^2. \quad \ldots \quad \ldots \quad (13)$$

We see, without difficulty that we may use either $\partial^2h/\partial p^2$ or $\partial^2h/\partial q^2$, and we find the two solutions

$$y=\tfrac{1}{4}e^x(\sin x-2x\cos x),$$
$$y=-\tfrac{1}{4}e^x(\sin x+2x\cos x),$$

each of which as a matter of fact includes a part usually absorbed into the complementary function.

Two comments are suggested by this last example. In the first place, we do not pretend to have here an exceptionally simple way of solving the equation (10); we are only concerned to shew that if the equation itself is contained under a more general form of which we have the solution, a solution of the equation is obtainable by a direct and unimpeachable process. And in the second place, instead of starting from a known result involving two parameters, we might have assumed that

$$(D^2-2D+2)\ F(x,\ p)=h(p)e^{px}\sin px, \quad \ldots \qquad \ldots\ (14)$$

where

$$F(x,\ p)=e^{px}(\sin px+p\cos px), \quad \ldots \qquad \ldots\ (15)$$

$$h(p)=-2(p-1)(p^2+1), \quad \ldots \quad \ldots \qquad \ldots\ (16)$$

and obtained the slightly different solution

$$y=-\tfrac{1}{4}e^x(\cos x+2x\cos x).$$

A similar possibility is clearly open in general. All that is required of $p_1, p_2, \ldots p_r$ is that they should be capable of taking simultaneously the values $a_1, a_2, \ldots a_r$, and subject to this condition they may be made from the commencement any convenient functions of a single parameter p. Thus if $a_1, a_2, \ldots a_r$ are all different from zero, we may put $p_k=a_kp$, and the particular equation to be studied will correspond to $p=1$, or in any case, we may put $p_k=a_k+p$ in the general equation which is given to us and consider what happens when $p=0$. But, since different equations may most naturally be discussed in different ways, there is evident advantage in limiting ourselves as little as possible in the general treatment.

(*To be continued.*)

Some Properties of Cyclic Equations.

By M. T. NARANIENGAR.

1. Consider the equation

$$f(x) \equiv x^2 + px + q = 0. \quad \ldots \quad \ldots \; (1)$$

The roots of this will be α, α^2, where $\alpha^4 \equiv \alpha$, if equation (1) is included in $f(x^2)=0$. Hence $f(x)$ must be a factor of $f(x^2)$.

Changing the sign of x, we find that $f(-x)$ is also a factor of $f(x^2)$.

Thus we are led to the identity

$$f(x^2) \equiv f(x)\, f(-x). \quad \ldots \quad \ldots \; (2)$$

Suppose, now that z is a quantity such that $z^2 = z$. ... ... (3)

Then substituting in (2) z for x and remembering that $z^2 = z$, we have, since $f(z) \neq 0$

$$1 = f(-z) \quad \ldots \quad \ldots \quad \ldots \; (4)$$

Equation (4) must be satisfied by the values of z found from (3).

Hence $$z^2 - pz + q - 1 \equiv z^2 - z.$$

In other words $p=1$, $q=1$; that is, the roots of the cyclic equation

$$f(x) \equiv x^2 + x + 1 = 0$$

are α, α^2, where $\alpha^4 \equiv \alpha$.

2. Next, suppose the roots of (1) are α, α^3, where $\alpha^9 \equiv \alpha$. In this case, we find that equation (1) is included in $f(y)=0$, where $y = x^3$; whence we obtain the identity

$$f(x^3) \equiv f(x) \cdot f(xw),\, f(xw^2) \quad \ldots \quad \ldots \quad \ldots \; (5)$$

where w^3 is unity.

Equation (5) leads to

$$p = p^3 - 3pq\,; \quad q = q^3\,;$$

whence $q = 0, \pm 1$; $p = 0, \pm 1, \pm 2, \pm\sqrt{-2}$.

Rejecting trivial and inadmissible values, we finally have the cylic quadratic equations

$$x^2 + 1 = 0,\; x^2 \pm x\sqrt{-2} - 1 = 0, \text{ whose roots are } \alpha,\, \alpha^3\; (\alpha^8 = 1).$$

3. Similarly, if α, α^4 be the roots, where $\alpha^{16} \equiv \alpha$, we have the identity

$$f(x^4) \equiv f(x).\; f(xw).\; f(xw^2).\; f(xw^3), \quad \ldots \quad \ldots \; (6)$$

where $w^4 = 1$.

Equation (6) results in the following relations:

$$p = 2q^2 - (p^2 - 2q)^2,$$
$$q = q^4\,;$$

whence we easily obtain the quadratics $x^2 + p_1 x + 1 = 0$, $x^2 + p_2 x + 1 = 0$; where $p_1\, p_2$ are the roots of $p^2 - p - 1 = 0$.

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In other words, $(x^2+p_1x+1)(x^2+p_2x+1)$ is a factor $(x^{15}-1)$. In fact, the product is $x^4+x^3+x^2+x+1=0$, so that the roots of the quadratics can be written in the form α, α^4; $\alpha^2, \alpha^3\ (=\alpha^8)$.

4. Generally, let the roots of (1) be α, α^n where $\alpha^{n^2}\equiv\alpha$. Then from the identity

$$f(x^n)\equiv f(x).\ f(xw).\ f(xw^2).\quad \ldots f(xw^{n-1}),\ \ldots \qquad \ldots (7)$$

where $w^n=1$, we deduce

$$p=(-1)^{n-1}\left\{p^n-n.\ p^{n-2}q+\frac{n.n-3}{1.2}.p\ p^{n-4}q^2-\right.$$
$$\left.\frac{n(n-4)(n-5)}{1.2.3}\,p^{n-6}q^3+-\ldots\right\}\ldots (8)$$

and

$$q=q^n.$$

Thus $q=\left(\cos\frac{2k\pi}{n-1}+i\sin\frac{2k\pi}{n-1}.\right)$ for all k; and the values of p are found from equation (8).

Now, equation (8) is always reducible. For since α, α^n are the roots of $x^2+px+q=0$,

$$q=\alpha^{n+1};\ p=-(\alpha+\alpha^n)=-(\alpha+q\alpha^{-1}).$$

$$\therefore \qquad \alpha=q^{\frac{1}{n+1}}\left(\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}\right),$$

and

$$\alpha^n=q\alpha^{-1}=q^{\frac{n}{n+1}}\left(\cos\frac{2kn\pi}{n+1}+i\sin\frac{2kn\pi}{n+1}\right)$$
$$=q^{\frac{1}{n+1}}\left(\cos\frac{2k\pi}{n+1}-i\sin\frac{2k\pi}{n+1}\right),$$

since $q^{n-1}=1$.

Hence

$$p=-2q^{\frac{1}{n+1}}\cos\frac{2k\pi}{n+1},$$

for all k, shewing that the values of p are the same for the pairs $(1, n)$. $(2, n-1)$, ... of values of k. In other words, the number of values of p cannot exceed $\frac{1}{2}(n+1)$.

Again, if we write $z=2\cos\frac{2k\pi}{n+1}$, we know that z satisfies an algebraic equation

$$\phi(z)=0,$$

which is of degree $\frac{1}{2}n$ or $\frac{1}{4}(n-2\pm1)$ according as n is even or odd,

Putting

$$p=-q^{\frac{1}{n+1}}z,$$

we deduce that equation (8) is reducible to

$$\phi\left(-p/q^{\frac{1}{n+1}}\right)=0,$$

which is of degree $\frac{1}{2}n$ or $\frac{1}{4}(n-2\pm1)$ according as n is even or odd.

The form of $\phi(z)$ may be obtained as follows:—

Suppose $\psi(m)\equiv z^{m}-m\,z^{m-2}+\frac{m(m-3)}{1\cdot2}\,z^{m-4}-\ldots\ldots;$

then

$$\phi(z)\equiv\psi\left(\frac{n}{2}\right)+\psi\left(\frac{n-2}{2}\right)+\ldots\ldots\ldots1=0,$$

if n is even;

and $$\phi(z)\equiv\psi\left(\frac{n-1}{4}\right)-\psi\left(\frac{n-5}{4}\right)+\ldots\ldots\ldots=0,$$

if n is odd and of the form $4\,m+1$; otherwise $\phi(z)\equiv\psi\left(\frac{n-1}{4}\right)$.

For, $\psi(m)=2\cos\frac{2\,mk\pi}{n+1}$, being the equivalent of $\lambda^{m}+\lambda^{-m}$, where $z=\lambda+\lambda^{-1}=2\cos\frac{2\,k\pi}{n+1}$; and

$$\phi(z)\equiv(\lambda^{\frac{1}{2}n}+\lambda^{--\frac{1}{2}n})+\lambda^{\frac{1}{2}(n-2)}+\lambda^{-\frac{1}{2}(n-2)}+\ldots\ldots$$

if n is even; and

$$\phi(z)=\lambda^{m}+\lambda^{-m}-\ldots\ldots\ldots$$

if n is $(4m+1)$, &c.

5. Consider, next, the cubic

$$f(x)\equiv x^{3}+px^{2}+qx+r=0. \quad\ldots\quad\ldots\quad(9)$$

If this should have roots α, α^{2}, α^{4}, where $\alpha^{8}=\alpha$, or $\alpha^{7}=1$, we write

$$f(x^{2})\equiv-f(x).\ f(-x)\text{ as in § 1.} \quad\ldots\quad\ldots\quad(10)$$

Giving values 0, 1 to x, we find $r=-1$, $p-q=1$; and by equating co-efficients of x^{4} in (10) $p^{2}+p-2q=0$. Thus, we have for p, the quadratic

$$p^{2}-p+2=0. \quad\ldots\quad\ldots\quad(11)$$

The equation for q is

$$q^{2}+q+2=0; \quad\ldots\quad\ldots\quad\ldots\quad(12)$$

whose roots differ from the roots of the former only in sign.

Hence, remembering that $p-q=1$, we deduce the following relations:

(i) If $p_1\,p_2$ are the roots of $p^{2}-p+2=0$, the cubic equation $x^{3}+p_1x^{2}-p_2x-1=0$ is cyclic and has for roots α, α^{2}, α^{4}, where $(\alpha^{7}=1)$.

(ii) Also

$$(x^{3}+p_1x^{2}-p_2x-1)(x^{3}+p_2x^{2}-p_1x-1)\equiv(x^{7}-1)/(x-1), \quad\ldots\quad(13)$$

the second factor $(x^3+p_2x^2-p_1x-1)$ corresponding to the roots α^3, α^6., α^5 $(=\alpha^{12})$.

From the trigonometrical expression for α *viz* $\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)$, we easily deduce that

$$\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)+i\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)=-p_1,$$

$$\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)-i\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}=\sin\frac{8\pi}{7}\right)=-p_2,$$

whence finally $$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\tfrac{1}{2}, \quad \ldots \quad \ldots (14)$$

$$\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\tfrac{1}{2}\sqrt{7}, \quad \ldots \quad \ldots (15)$$

by virtue of equation (11).

6. If the roots of the cubic in the last article should be α, α^3, α^9 where $\alpha^{26}=1$, we obtain, as in § 2,

$$f(x^3)\equiv f(x).\, f(xw).\, f(xw^2). \quad [w^3=1].$$

Equating co-efficients

$$\left.\begin{aligned} p&=p^3-3pq+3r,\\ q&=q^3-3pqr+3r^2,\\ r&=r^3.\end{aligned}\right\} \quad \ldots \quad \ldots (16)$$

Hence if $r=1$, p, q are found from

$$p=p^3-3pq+3$$
$$q=q^3-3pq+3.$$

Rejecting the case $p=q$,

$$p^2+pq+q^2=1,\ p+q=p^3+q^3-6pq+6.$$

Put $p+q=a$, $pq=b$, then

$$a^2-b=1,\ a=a(1-2b)-6b+6.$$

Hence, eliminating b,

$$2a^3+6a^2-2a-12=0.$$

$$\therefore \quad (a+2)(a^2+a-3)=0 \ \ldots \quad \ldots \quad \ldots (17)$$

$$\therefore \quad a=-2,\ b=3\ ;\ \text{or}\ a=-\tfrac{1}{2}(1\pm\sqrt{13}),\ b=\tfrac{1}{2}(5\pm\sqrt{13}).$$

The first set of values determines p, q as the roots of the quadratic $\lambda^2+2\lambda+3=0$; and the corresponding cubic x^3+px^2+qx+1 is divisible by $x-1$, since the sum of the co-efficients is $p+q+2=0$. In this case, therefore, the property investigated relates to the resulting quadratic $x^2+(1+\lambda)x-1=0$, and is readily seen to be identical with that stated in § 2.

Taking a as a root of $a^2+a-3=0$ and $b=a^2-1=2-a$, we have the following:—

If λ, λ_2 are the roots of $\lambda^2-a\lambda+b=0$, the equation $x^3+\lambda_1x^2+\lambda_2x+1=0$ is cyclic and will have for roots α, α^3, α^9. $[\alpha^{13}=-1]$.

Similarly, if $r=-1$, we find that the cubic

$$x^3-\lambda_1 x^2+\lambda_2 x-1=0$$

will have roots α, α^3, α^9, where $\alpha^{13}=1$.

As in § 5, writing $\alpha=\cos\dfrac{2\pi}{13}+i\sin\dfrac{2\pi}{13}$, we deduce

$$\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{18\pi}{13}=\tfrac{1}{2}\,a, \qquad \ldots \qquad \ldots \text{ (18)}$$

$$\sin\frac{2\pi}{13}+\sin\frac{6\pi}{13}+\sin\frac{18\pi}{13}=\tfrac{1}{2}\,\sqrt{(5-3a)}. \qquad \ldots \text{ (19)}$$

7. In the examples considered in §§ 5 and 6, we might have proceeded as follows:—

Suppose the cubic is $x^3-p\,x^2+qx-r=0$, and that the roots are α, α^2, α^4, $(\alpha^7=1)$.

Then we have

$$\begin{aligned} p&=\alpha+\alpha^2+\alpha^4\\ q&=\alpha.\alpha^2+\alpha^2.\alpha^4+\alpha^4.\alpha\\ &=\alpha^3+\alpha^6+\alpha^5,\\ r&=\alpha.\alpha^2.\alpha^4=\alpha^7=1. \end{aligned}$$

$\therefore$ $$p+q=\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=-1, \text{ as is well-known;}$$

and $$\begin{aligned} p.q&=(\alpha+\alpha^2+\alpha^4)(\alpha^3+\alpha^6+\alpha^5)\\ &=\alpha^4+\alpha^7+\alpha^6+\alpha^6+\alpha^8+\alpha^7+\alpha^{10}+\alpha^9\\ &=3+q+p, \text{ since } \alpha^7=1. \end{aligned}$$

Hence p, q are the roots of the quadratic equation

$$\lambda^2+\lambda+2=0.$$

In other words, if λ_1, λ_2 are the roots of this quadratic, the roots of the cubic

$$x^3-\lambda_1 x^2+\lambda^2 x-1=0$$

are α, α^2, α^4, where $\alpha^7=1$.

Again, let the roots be α, α^3, λ^9. $[\alpha^{13}=1]$. Then

$$\begin{aligned} p&=\alpha+\alpha^3+\alpha^9\\ q&=\alpha.\alpha^3+\alpha^3.\alpha^9+\alpha^9\alpha\\ &=\alpha^4+\alpha^{12}+\alpha^{10}\\ r&=\alpha.\alpha^3.\alpha^9=\alpha^{13}=1. \end{aligned}$$

$\therefore$ $$\begin{aligned} p^2&=\alpha^2+\alpha^6+\alpha^5+2\,(\alpha^4+\alpha^{12}+\alpha^{10})\\ q^2&=\alpha^8+\alpha^{11}+\alpha^7+2\,(\alpha^3+\alpha^9+\alpha) \end{aligned}$$

∴ $$p^2+q^2=-1+p+q, \text{ since } \sum_{k=1}^{12}(\alpha^k)=-1;$$

and $$pq=\alpha^5+1+\alpha^{11}+\alpha^7+\alpha^3+1+1+\alpha^8+\alpha^6$$
$$=3-1-p-q.=2-p-q.$$

∴ $$p^2+q^2+pq=1.$$

Putting $p+q=a$, $pq=b$, we find
$$a^2-b=1;\ a+b=2.$$

∴ $$a^2+a-3=0.$$

8. We shall now consider the general irreducible cyclic equation of the third degree, *viz.*

$$f(x)\equiv x^3+px^2+qx+r=0.$$

Let the roots α_1 α_2 α_3 of this equation be connected by the relations

$$\alpha_2=\phi(\alpha_1), \alpha_3=\phi(\alpha_2), \alpha_1=\phi(\alpha_3).$$

By the Tschirnhausen transformation we know that the most general form for ϕ is of the second degree, so that we may write

$$\phi(\alpha)\equiv a\alpha^2+b\alpha+c=a(x-h)(x-k) \text{ say.} \quad \ldots \quad (20)$$

Hence $f(x)$ must be a factor of $f[ax^2+bx+c]$.

Replacing x by $(h+k-x)$, we see that

$$ax^2+bx+c \text{ or } a(x-h)(x-k)$$

remains unchanged, and therefore

$f(-b/a-x)$ or $f(h+k-x)$ is also a factor of $f[ax^2+bx+c]$.

Thus we have the identity

$$f[ax^2+bx+c]\equiv-a^3f(x).\ f(-b/a-x). \quad \ldots \quad \ldots \quad (21)$$

Putting $x=\lambda$, μ in (21) where λ, μ are the roots of

$$ax^2+bx+c=x, \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

we get two linear relations among the co-efficients p, q, r; *viz:*

$$1=-a^3f(-b/a-\lambda),$$
$$1=-a^3f(-b/a-\mu).$$

In fact, the expression $(ax^2+bx+c-x)$ is a factor of

$$\{1+a^3.\ f(-b/a-x)\}.$$

Equating co-efficients of x^4 in (21) a third relation is obtained whence the values of p, q, r easily follow.

[Cajori's *Theory of Equations*, § 176.]

9. Question 507 of Mr. S. Ramanujan is an example of the above method. [J. I. M. S., Vol. VI, No. 2, p. 74.]

To solve the equations $x^2=y+a$, $y^2=z+a$, $z^2=x+a$; we write

$$f(x)\equiv x^3+px^2+qx+r,$$

and deduce, as in § 8,

$$f(x^2-a)\equiv-f(x).\ f(-x) \ldots \quad \ldots \quad \ldots$$

The roots of

$$x^2-x-a=0 \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

are $\frac{1}{2}(1\pm\sqrt{1+4a})$. Call them λ, μ; so that

$$\lambda+\mu=1,\ \lambda\mu=-a. \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Put $x=\lambda$, μ in (23), and we have

$$\left.\begin{array}{l} 1=\lambda^3-p\lambda^2+q\lambda-r, \\ 1=\mu^3-p\mu^2+q\mu-r. \end{array}\right\} \quad \ldots \quad \ldots \quad (26)$$

$\therefore$ $$0=\lambda^2+\lambda\mu+\mu^2-p\,(\lambda+\mu)+q$$

i.e., $$0=1+a-p+q, \text{ using (25)} \quad \ldots \quad \ldots \quad (27)$$

Also, eliminating q from (26)

$$a-1-pa-r=0. \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

The third equation connecting p, q, r, is got from the constant term of (23), *viz*:

$$r^2+r-a^3+pa^2-qa=0. \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

Equations (27), (28), (29) will give two sets of values for p, q, r corresponding to the two cubic factors of the sextic

$$[\,\{\,x^2-a)^2-a\,\}^2-x-a]\div(x^2-x-a)$$

10. As a last example, we shall investigate a cyclic cubic equation whose roots are connected by the following relations:

$$\alpha_2=\alpha_1^2-2\alpha_1,\ \alpha_3^2=\alpha_2^2-2\alpha_2,\ \alpha_1^2=\alpha_3^2-2\alpha_3.$$

Our method leads to the identity

$$f(x^2-2x)\equiv-f(x).\ f(2-x). \quad \ldots \quad \ldots \quad (30)$$

Putting $x=0$, 3, we deduce

$$1=-(8+4p+2q+r)$$
$$1=-(-1+p-q+r).$$

$\therefore$ $$p+q+3=0 \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

Also, equating co-efficients of x^4 in (30)

$$12+p=12+4p+q+p(-6-p)+q.$$

$\therefore$ $$p^2+3p+2q=0. \quad \ldots \quad \ldots \quad (32)$$

From (31) and (32)

$$p^2+5p+6=0,$$

or $$(p+2)(p+3)=0.$$

Hence

$$p=-2,\ q=-1,\ r=1;$$
$$p=-3,\ q=0,\ r=3.$$

In other words, the cubic equations

$$x^3-2x^2-x+1=0,$$
$$x^3-3x^2+3=0,$$

are cyclic, the relations connecting the roots being

$$\alpha_2=\alpha_1^2-2\alpha_1,\ \alpha_3^2=\alpha_2^2-2\alpha_2,\ \alpha_1^2=\alpha_3^2-2\alpha_3.$$

SHORT NOTES.

A note on $\int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2}dx$.

The value of this integral is usually given as $\frac{\pi}{b}\log\frac{a+b}{b}$, but this result is affected by a change in the sign of a or b, whereas the integrand is independent of the sign of a or b. The present note is intended to explain this apparent fallacy.

Let
$$I=\int_{-\infty}^{+\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2}dx;$$

so that
$$\frac{\partial I}{\partial a}=2a\int_{-\infty}^{+\infty} \frac{x^2}{(1+a^2x^2)(1+b^2x^2)}dx.$$

This can now be integrated by the methods of partial fractions.

But the correct result is obtained by integrating round a suitable contour.

The function $\frac{z^2}{(1+a^2z^2)(1+b^2z^2)}$ has four simple poles, *viz*:

$$z=\pm\frac{i}{b},\ z=\pm\frac{i}{b}.$$

Take a contour on the upper half of the z—plane consisting of the real axis from $-\infty$ to $+\infty$, and a "large" semi-circle Γ.

(1) If a and b are both positive, the only poles of the function within the contour are $+\frac{i}{a}$, $+\frac{i}{b}$.

(2) If a and b are both negative, then the poles of the function within the contour are $-\frac{i}{a}$, $-\frac{i}{b}$.

In any case the two poles on the upper half of the z—plane are

$$\frac{i}{a_1}\ \frac{i}{b_1}$$

and $I=2\pi i\times$ sum of the residues at the poles $\frac{i}{a_1}$, $\frac{i}{b_1}$,

where a_1, b_1, the numerical values of for a, b.

By the usual method we find

$$\text{sum of the residues}=\frac{1}{i\,a_1\,b_1\,(a_1+b_1)}.$$

Now $\int_C=\int_\Gamma+\int_{-\infty}^{+\infty}$. At all points on Γ the integrand is sensibly equal to $\frac{1}{a^2b^2z^2}$ which vanishes in the limit when the radius of Γ is sufficiently large.

$$\therefore \qquad \frac{\partial I}{\partial a} = \int_{\infty}^{\infty} \frac{x^2}{(1+a^2x^2)(1+b^2x^2)} dx = 2\pi i \times \frac{1}{i\, a_1 b_1 (a_1+b_1)}.$$

$$\therefore \qquad I = \frac{2\pi}{b_1} \log \frac{a_1+b_1}{b^1}.$$

and the required integral is $\frac{\pi}{b_1} \log \frac{a_1+b_1}{b_1}$,
which is the correct form of the result.

The results

$$\int_{\circ}^{\frac{1}{2}\pi} \log\,(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\, d\theta = \pi \log \tfrac{1}{2}\,(\alpha+\beta)$$

and
$$\int_{\circ}^{\pi} \frac{\cos 2\,rx}{1-2a\cos x+a^2} = \frac{\pi a^{2r}}{1-a^2}$$
can also be similarly explained

The discussion given here is probably nothing new, but the fact that it is not explained in some of the well-known text-books on Calculus is perhaps a sufficient justification for this note.

J. M. Bose.

The Face of the Sky for March and April 1915.

Sidereal time at 8 p.m.

D.	*March.* H.	M.	S.	*April.* H.	M.	S.
1	6	33	14	8	35	27
8	7	0	49	9	3	3
15	7	28	25	9	30	39
22	7	55	1	9	58	14
29	8	22	37	50	25	50

From this table the constellations visible during the evenings can be ascertained by a reference to their positions as given in the star-chart,

Phases of the Moon.

	March.				*April.*			
	D.	H.	M.		D.	H.	M.	
Full Moon	2	0	3	A.M.				
Last Quarter	8	5	58	P.M.	7	1	42	A.M.
New Moon	16	1	12	A.M.	14	5	6	P.M.
First Quarter	24	4	18	„	22	9	9	„
Full Moon	31	11	8	„	29	7	49	„

It may be noticed that January and March have each two Full Moons and February is without a Full Moon.

The Sun

enters the Vernal Equinox on March 21 at 10-21 A.M.

The Planets.

Mercury is stationary on March 6 and attains its greatest elongation (27° 43′ W) on March 20. It is in conjunction with the Moon on March 13 and April 13, with Jupiter on March 30 and with Mars on April 4.

Venus is in conjunction with the Moon on March 12 and April 11, with Uranus on March 19 and with Jupiter on April 15. It is near the boundary between Capricornus and Aquarius in March and in Pisces in April.

Mars is in conjunction with the Moon on March 14 and April 12 and with Jupiter on March 24. It is in Pisces in March and in Aries in April.

Jupiter is in conjunction with the Moon on March 14 and April 11. It is in Pisces in these months.

Saturn is in quadrature to the Sun on March 17. It is in conjunction with the Moon on March 23 and April 20.

Uranus is in conjunction with the Moon on March 12 and April 9.

Neptune is in quadrature to the Sun on April 19 and is in conjunction with the Moon on March 26 and April 22.

V. RAMESAM.

SOLUTIONS.

Question 548.

(A. Narasinga Rao):—If $u_r = \dfrac{x^r}{1-x^r}$, shew that the infinite series

$$u_1+u_2+u_3+u_5-u_6+u_7-u_{10}\ldots\ldots = x\,\frac{1+x}{1-x},$$

where the suffixes of u are the numbers having non-repeated prime factors, the terms being taken with a $+$ *ve* or $-$ *ve* sign according as the number of such factors is odd or even.

Solution by T. P. Trivedi, M.A., LL.B., N. Durairajan,
R. Vythynathaswamy and N. Sankara Aiyar.

Let us find the coefficient of x^N by picking out the terms in which it occurs.

Let $N = a^\alpha b^\beta c^\gamma \ldots\ldots$, where a, b, c are the prime factors of N, which are n in number.

Then x^N occurs in u_1 which is $1+x+x^2\ldots\ldots\ x^N+\ldots+\ldots$

It occurs also

once with the $+$ sign in each of the terms ua, ub, uc, ...,

once with the $-$ sign in each of $-uab$, $-uac$, $-uad$, ...,

once with the $+$sign in each of $uabc$, $uabd$, ...,

and so on. Hence the coefficient of x^N

$$\begin{aligned}
&= 1+n-\tfrac{1}{2}\,n\,(n-1)+\frac{1}{3\,!}\,n\,(n-1)\,(n-2)-\ldots\\
&= 2-\left\{1-n+\frac{n\,(n-1)}{1\cdot2}-\frac{n\,(n-1)\,(n-2)}{1\cdot2\cdot3}+\ldots\right\}\\
&= 2-(1-1)^n\\
&= 2.
\end{aligned}$$

The coefficient of x is 1.

Thus the given series $=x+2x^2+2x^3+\ldots$

$$\begin{aligned}
&= x+\frac{2x^2}{1-x}=\frac{x+x^2}{1-x}\\
&= x\,\frac{1+x}{1-x}.
\end{aligned}$$

SRI PRATAP

Question 550.

(K. J. SANJANA) :—At a game of bridge recently played it was found that every player had three cards of each of three suits and four cards of the remaining suit; what is the chance of this distribution of the cards ?

Solution by T. P. Trivedi, M. A., LL.B., and A. Narasinga Rao.

It is obvious that the 52 cards can be distributed amongst all the four players in the most general manner in $\dfrac{\underline{|52}}{\{\underline{|13}\}^4}$.

According to the data in the question, the first player may get his cards in ${}^4C_1\times{}^{13}C_3\times{}^{13}C_3\times{}^{13}C_3\times{}^{13}C_4$, ways, since he may have 4 cards of any of the four suits; the second can now get his cards in ${}^3C_1\times{}^{10}C_3\times{}^{10}C_3\times{}^9C_3\times{}^{10}C_4$ ways; the third player may have his cards in ${}^2C_1\times{}^7C_3\times{}^6C_3+{}^6C_3\times{}^7C_4$ ways; the rest of the cards go to the last player. The number of ways of distributing the cards is thus

$$\frac{\underline{|4}\,(\underline{|13})^4}{(\underline{|3})^{12}(\underline{|4})^4},$$ by multiplying the above result.

Hence the chance of this distribution is

$$\frac{(\underline{|13})^8}{(\underline{|3})^{12}(\underline{|4})^3\,\underline{|52}}.$$

Question 552.

(S. KRISHNASWAMI AIYANGAR) :—Find the sum of

$$\frac{2}{3}x+\frac{1}{3\cdot5\cdot2}\,\frac{x^2}{1\,!}+\frac{1\cdot3}{5.7\cdot2^3}\,\frac{x^3}{2!}+\frac{1\cdot3\cdot5}{7\cdot9\cdot2^5}\cdot\frac{x^4}{3\,!}+\ldots\ldots$$

$$+\frac{1\cdot3\cdot5\ldots(2n-1)}{(2n+1)(2n+3)}\cdot\frac{1}{2^{2n-1}}\cdot\frac{x^{n+1}}{n\,!}+\ldots\ldots$$

Solution by T. P. Trivedi, M.A., LL. B. and K. J. Sanjana, M.A.

The general term is

$$\frac{1\cdot3\cdot5\cdot\ldots\ldots(2n-\!-1)}{(2n+1)(2n+3)}\cdot\frac{1}{2^{2n-1}}\cdot\frac{x^{n+1}}{\underline{n!}}$$

$$=\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots\ldots2n}\cdot\frac{1}{2n+1}\,\frac{x^{n+\frac{3}{2}}}{2n+3}\cdot\frac{1}{x^{\frac{1}{2}}2^{n-1}}$$

$$=\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\,\frac{2}{2n+3}\cdot\frac{x^{n+\frac{3}{2}}}{2^{n+\frac{1}{2}}}\,\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\cdot\frac{1}{2n+1}$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\cdot\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\int_0^x\frac{x^{n+\frac{1}{2}}}{+\;\frac{1}{2}}dx\cdot\frac{1}{2n+1}$$

Now, since $\sin^{-1}z=\Sigma\,\frac{1\cdot3\cdot5\ldots2n-1}{2\cdot4\cdot6\ldots2n}\,\frac{z^{2n+1}}{2n+1}$, the summation being confined to odd powers of z, the above expression is equal to

$$\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\int_0^x\sin^{-1}\sqrt{\frac{x}{2}}dx$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\left\{x\sin^{-1}\sqrt{\frac{x}{2}}-\int\frac{x}{\sqrt{1-\frac{x}{2}}}\cdot\frac{1}{2\sqrt{2x}}dx\right\}$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\left\{x\sin^{-1}\sqrt{\frac{x}{2}}-\frac{1}{2}\int\frac{x\,dx}{\sqrt{2x-x^2}}\right\}$$

$$=\frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}}\left\{x\sin^{-1}\sqrt{\frac{x}{2}}+\tfrac{1}{2}\sqrt{2x-x^2}-\tfrac{1}{2}\operatorname{vers}^{-1}x\right\},$$

no constant being added, since the expression as well as the integral vanish with x.

Question 553.

(S. Krishnaswami Aiyangar) :—ABC is a triangle such that its sides subtend equal angles at one focus S of an inscribed conic. With S as focus four more conics are drawn circumscribing the triangle. Show that

(i) The points of contact with the sides of the inscribed conic are on the focal radii through the vertices of the triangle.

(ii) The directrix of one of the circumconics is identical with the directrix of the inscribed conic corresponding to S.

(iii) The directrix of one of the other circumconics and its tangent through C meet at the point of intersection of AB and the directrix of the in-conic.

(iv) If this point is P_3, then P_3SC' is a right angle, where C′ is the point of contact of the inscribed conic with the side AB.

Solution by R. J. Pocock and R. Vythynathaswamy.

Since tangents, subtend equal angles at the focus and AB, BC, CA also subtend equal angles at S, it follows that the angles BSC′, C′SA

ASB′ are all equal to 60° and therefore **BSB′ is a straight line, which** proves (i).

Let the bisector of ∠BSA′ meet AB in P_3. Then $P_3SC'=90°$ which proves (4), since P_3 will now be seen to be identical with P_3 of the question.

If CC′ cut the inscribed conic in F, P_3F is a tangent and since FC′ is a focal chord, P_3 lies on the directrix of the inscribed conic; similarly for P_2.

But since P_3S and P_2S bisect BSA′, and BSC′, P_2,P_3 lie on the directrix of the circumconic, which proves (ii).

Also since P_3S bisects ∠BSA′ and $\angle P_3SC=90°$, P_3 lies on the directrix of the circum-conic. Thus P_3C is tangent at C, which proves (iii).

Question 554.

(T. P. Trivedi, M. A., LL. B.):—If V denote the entire volume of the solid $\left(\frac{x}{a}\right)^n+\left(\frac{y}{2}\right)^n+\left(\frac{z}{c}\right)^n=1$, and A the whole area of its trace on the xy—plane, prove that

$$\frac{\text{V}}{\text{A}}=\frac{4c}{3n}\cdot\quad B\left(\frac{2}{n},\frac{1}{n}\right).$$

and deduce that for the solid

$$\left(\frac{x}{a}\right)^{\frac{2p}{2p+1}}+\left(\frac{y}{b}\right)^{\frac{2p}{2p+1}}+\left(\frac{z}{c}\right)^{\frac{2p}{2p+1}}=1$$

$$\frac{\text{V}}{\text{A}}=\frac{4c}{9p}\cdot\frac{(p+1)(2p+1)}{(2p+3)(4p+3)}\cdot\quad B\left(\frac{1}{2p},\frac{1}{p}\right).$$

Solution by R. J. Pocock, B.A.,B. Sc., F. R. A. S. and
R. Vythynathaswamy.

As V, A are both Dirichlet Integrals, by the ordinary theorems

$$\text{V}=\frac{8abc}{n^3}\cdot\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^3}{\Gamma\left(\frac{3}{n}\right)}\cdot\int_0^1 x^{\frac{3}{n}-1}\,dx=\frac{8abc}{3n^2}\frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}}{\Gamma\left(\frac{3}{n}\right)}\cdot$$

$$\text{A}=\frac{4ab}{n^2}\cdot\frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}^2}{\Gamma\left(\frac{2}{n}\right)}\cdot\int_0^1\left\{x^{\frac{2}{n}-1}\,dx\right\}=\frac{4ab}{2n}\frac{\left\{\Gamma\left(\frac{1}{n}\right)\right\}}{\Gamma\left(\frac{2}{n}\right)}\cdot$$

$$\therefore\quad\frac{\text{V}}{\text{A}}=\frac{4c}{3n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{3}{n}\right)}=\frac{4c}{3n}\cdot\quad B\left(\frac{2}{n},\frac{1}{n}\right).$$

Put $n=\frac{2p}{2p+1}$,

$$\therefore \qquad \frac{V}{A}=\frac{4c}{6p}(2p+1).\frac{\Gamma\left(1+\frac{1}{2p}\right)\Gamma\left(+\frac{2}{2p}\right)}{\Gamma\left(3+\frac{3}{2p}\right)}$$

$$=\frac{8c}{9p}.\ \frac{(p+1)(2p+1)}{(2p+3)(4p+3)}.\ B\left(\frac{1}{p},\frac{1}{2p}\right).$$

Question 555.

(R. VYTHYNATHASWAMY) :—P is a point on one of a system of curves $f(r_1, r_2, \alpha)=0$ where α is a variable parameter. If dn is the distance at the point P between this curve and its consecutive whose parameter is $\alpha+d\alpha$, show that

$$\frac{dn}{d\alpha}=\frac{1}{PQ}.\frac{\partial f}{\partial \alpha},$$

PQ being the diagonal of a parallelogram whose adjacent sides are along r_1, r_2, and equal to $\frac{\partial f}{\partial r_1}$ and $\frac{\partial f}{\partial r_2}$ respectively.

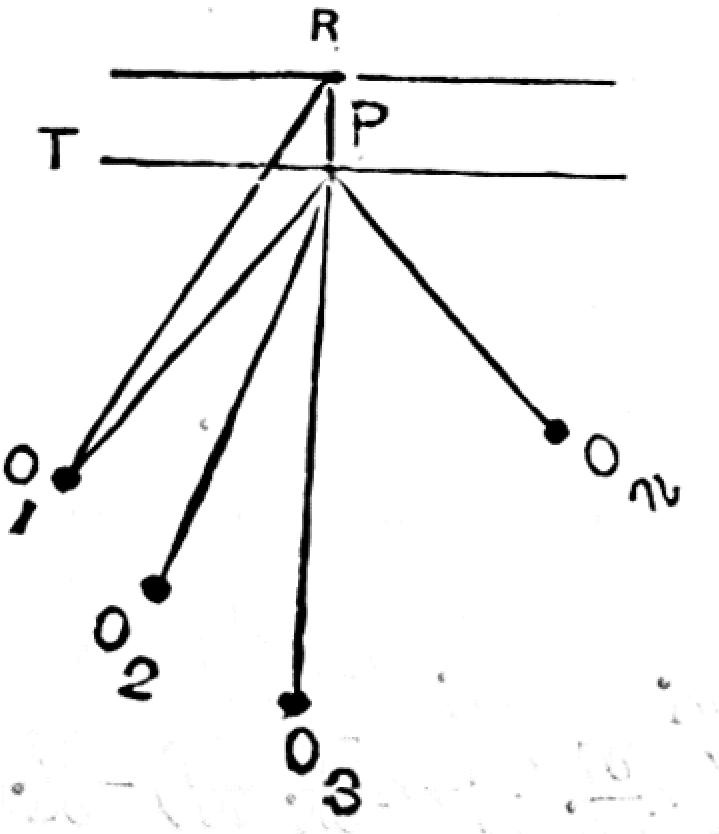
Solution (1) by N. Durairajan ; (2) by T. P. Bhaskarasastri M. K. Kevalramani and A. Narasinga Rao.

(1) The theorem may be proved for the general case as follows :—

Let PT be the tangent at P to the curve $f\ (r_1\ r_2 \ldots r_n,\ \alpha)=0$; and let the $\angle$T PO$_n=\alpha_n$.

Differentiating with respect to the arc, we have

$$\frac{\partial f}{\partial r_1}.\frac{dr_1}{ds}+\cdots\frac{\partial f}{\partial r_n}.\frac{dr_n}{ds}=0.$$



i.e. $f_1\cos\alpha_1 + \dots f_n\cdot\cos\alpha_n = 0,$ ... ... (i)

where f_m denotes $\partial f/\delta r_m$.

Now, the distance between a curve and its consecutive is the common normal to the two at any selected point on the first.

Hence, if we measure a distance dn equal to PR on the normal at P outwards, R will be the point on the consecutive curve.

The distances of R from the poles are

$$r_1 + dn\,.\,\sin\alpha_1,\quad r_2 + dn\,.\,\sin\alpha_2,\ \text{etc.}$$

which must satisfy the equation $f(r_1\, r_2 \dots\dots r_n,\ \alpha + d\alpha) = 0$.

$$\therefore\quad f(r_1 + dn\ \sin\alpha_1 \dots\dots\dots\dots\dots\dots, \alpha + d\alpha) = 0.$$

$$\therefore\quad f(r_1 \dots\dots \alpha) + \frac{\partial f}{\partial r_1}\cdot dn\ \sin\alpha_1 + \dots\dots + \frac{\partial f}{\partial\alpha}\cdot d\alpha) + \dots\dots = 0.$$

$$\therefore\quad \frac{\partial f}{\partial r_1}\cdot dn\ \sin\alpha_1 + \dots\dots\dots \frac{\partial f}{\partial\alpha}.d\alpha = 0,$$

since $f(r_1, r_2 \dots\dots \alpha) = 0$.

Thus $f_1\ \sin\alpha_1 + \dots\dots\dots\dots\dots\dots = -\frac{d\alpha}{dn}\cdot\frac{\partial f}{\partial\alpha}.$ ... (ii)

Squaring and adding the relations (i) and (ii), we get

$$(f_1\cos\alpha + \dots)^2 + (f_1\sin\alpha_1 + \dots)^2 = \left(\frac{d\alpha}{dn}\right)^2\left(\frac{\partial f}{\partial\alpha}\right)^2.$$

i.e., $f_1^2 + \dots f_n^2 + \sum 2.\, f_1\, f_2\cdot\cos(\alpha_1 - \alpha_2) = \left(\frac{d\alpha}{dn}\right)^2\left(\frac{\partial f}{\partial\alpha}\right)^2.$

Also, if PQ is the resultant of forces $f_1 f_2 \dots f_n$ at P along $r_1, r_2 \dots r_n$,

$$PQ^2 = \Sigma(f_1^2) + 2\Sigma f_1 f_2\cos(\alpha_1 - \alpha_2).$$

Hence $\frac{dn}{d\alpha} = \frac{1}{PQ}\ \frac{\partial f}{\partial\alpha}.$

(2) It is proved in Williamson, *Diff. Calculus*, p. 234, § 193, that PQ is normal to the curve at P.

Let r_1, r_2 be the bipolar coordinates of P, and $r_1 + dr_1$, $r_2 + dr_2$, those of R, where the normal cuts the consecutive curve. Then $PR = dn$.

Now $dr_1 =$ projection of dn on radius vector $= -dn.\ r_1\frac{d\theta_1}{ds}$.

Similarly $dr_2 = -dn.\ r_2\cdot\frac{d\vartheta}{ds}.$

Since R is a point on the consecutive curve whose parameter is $\alpha + d\alpha$, we have

$$\frac{\partial f}{\partial r_1}\cdot dr_1 + \frac{\partial f}{dr_2}\cdot dr_2 + \frac{\partial f}{\partial\alpha}d\alpha = 0.$$

$$\therefore\quad dn\left(r_1\frac{\partial f}{\partial r_1}\cdot\frac{d\theta_1}{ds} + r_2\frac{\partial f}{\partial r_2}\cdot\frac{d\theta_2}{ds}\right) = \frac{\partial f}{\partial\alpha}\cdot d\alpha.$$

But $$PQ=r_1\frac{\partial f}{\partial r_1}\cdot\frac{d\theta_1}{ds}+r_2\frac{\partial f}{\partial r_2}\cdot\frac{d\theta_2}{ds}$$

∴ $$PQ.\ dn=\frac{\partial f}{\partial \alpha}\cdot d\alpha.$$

∴ $$\frac{dn}{d\alpha}=\frac{1}{PQ}\cdot\frac{\partial f}{\partial \alpha}.$$

Question 556.

(R. Vythynathasawmy) :—Shew that in a cissoid the product of the perpendicular from the cusp on a tangent and the length of the tangent between the point of contact and the asymptote varies as the ordinate of the point of contact.

Solution by A. V. K. Krishna Menon, M.A., B.L., L.T. and several others.

The equation of the cissoid is $y^2\,(2a-x)=x^3$.

Let $(x,\ y)$ be the point of contact.

If ψ be the angle the tangent make with the x axis and t the length of the tangent between the point of contact and the asymptote, we have

$$t=(2a-x)\sec\psi.$$

Also, if p be the perpendicular from the cusp on the tangent

$$p=\left(x\frac{dy}{dx}-y\right)\cos\psi.$$

Now $$y\frac{dy}{dx}=\frac{x^2\,(3a-x)}{(2a-x)^2}$$

∴ $$xy\frac{dy}{dx}-y^2=\frac{x^3\,(3a-x)}{(2a-x)^2}-\frac{x^3}{2a-x}=\frac{ax^3}{(2a-x)^2}$$

∴ $$tpy=(2a-x)\left(xy\frac{dy}{dx}-y^2\right)=(2a-x)\frac{ax^3}{(2a-x)^2}$$

$$=\frac{ax^3}{2a-x}=ay^2$$

Hence $$tp=ay,$$

or $$tp\propto y.$$

Question 557.

(Professor Sanjana) :—At a point P of a parabola the norma chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that PQ. AN=2 SP. PG.

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$SP_1.SP_2=AS.SP$, $PP_1.PP_2=PQ.AN$, where S is the focus.

Shew also that the circle P_1P_2S passes through the circumcentre of PP_1P_2.

Solution by A. V. K. Krishna Menon, M.A., B.L.,L.T.; R. Vythynathaswamy, V. D. Gokhale, V. V. S. Narayanan, D. Krishnamurti and L. N. Subramaniam, M.A., L.T.

(i) Let P be the point $(at^2, 2\,at.)$ The equation of PQ is $tx+y=at^3+2at$. Where this meets the parabola $y^2t+4ay=4a^2t^3+8a^2t$.

i.e. $(y-2at)\,[yt+(2at^2+4a)]=0$.

∴ If y', y'' be the ordinates of P and Q,

$$\frac{PG}{QG}=\frac{y'}{y''}=\frac{2at}{(2at^2+4)/t}=\frac{2at^2}{2at^2+4a}$$

$$=\frac{at^2}{at^2+2a}=\frac{AN}{AG},$$ with the usual notation.

$$\therefore \qquad \frac{PG}{PQ}=\frac{AN}{AN+AG}=\frac{AN}{2(a+AN)}=\frac{AN}{2SP}$$

$$\therefore \qquad PQ.\ AN=2SP.\ PQ.$$

This proves the first part.

(ii) Now equation of the curve passing through the feet of the normals drawn from P $(at^2, 2at)$ is

$$(at^2-x)y+(2at-y)2a=0.$$

Where this meets theparabola

$$(4at^2t^2-4y^2)y+(8a^2t-4ay)2a=0$$

ie. $$y^3+4a^2y(2-t^2)-16a^3t=0$$

ie. $$(y-2at)(y^2+2aty+8a^2)=0.$$

If h_1 and y_2 are the ordinates of P_1 and P_2, they are the roots of the equation $y^2+2aty+8a^2=0$.

If x_1 and x_2 are the abscissæ of P_1 and P_2

$$SP_1.\ SP_2=(a+x_1)(a+x_2)$$

$$=a^2+a(x_1+x_2)+x_1x_2$$

$$=a^2+\tfrac{1}{4}(y_1^2+y_2^2)+\frac{1}{16a^2}y_1^2y_2^2$$

$$=a^2+\tfrac{1}{4}[4a^2t^2-16a^2]+4a^2,$$ since $y_1+y_2=2at$ and $y_1y_2=8a^2$,

$$=a(at^2+a)=a(AN+a)=AS.\ SP.$$

Again, if PP_1 and PP_2 cut the axis in G_1 and G_2

$PP_1.\ x_1=2SP_1.\ PG_1$, and $PP_2.\ x_2=2SP_2.\ P_2G_2$ [using the 1st part].

∴ $PP_1.\ PP_2.\ x_1x_2=4SP_1.\ SP_2.\ P_1G_1.\ P_2G_2$ ... ... (1)

Now $x_1x_2=\dfrac{y_1y_2}{16a^2}=4a^2$ ... ... (2)

$SP_1.\ SP_2=AS.\ SP$ ... ... (3)

$$P_1G_1.\ P_2G_2=\sqrt{(4a^2+y_1^2)(4a^2+y_2^2)}$$

$$=\sqrt{16a^4+4a^2(4a^2t^2-16a^2)+64a^4}$$

$$=\sqrt{16a^4+16a^4t^2}$$
$$=2a\sqrt{4a^2+4a^2t^2}=2a\sqrt{PN^2+NG^2}$$
$$=2a.\ PG\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ (4)$$

From (1), (2), (3) and (4), we get

$$PP_1.\ PP_2=2SP.\ PG$$
$$=PQ.\ AN \quad \text{using (i).}$$

This proves the second part.

(iii) $SG_1=SP_1$; $SG_2=SP_2$.

$\therefore \qquad M\hat{S}P_1=\pi=2\theta,\ \ M\hat{S}P_2=\pi-2\theta_2.$

$\therefore \qquad P_1\hat{S}P_2=2(\theta_2-\theta_1)=2\ P_1SP_2$

=the angle subtended by P_1P_2 at the circumcentre of $\Delta\ P_1PP_2$.

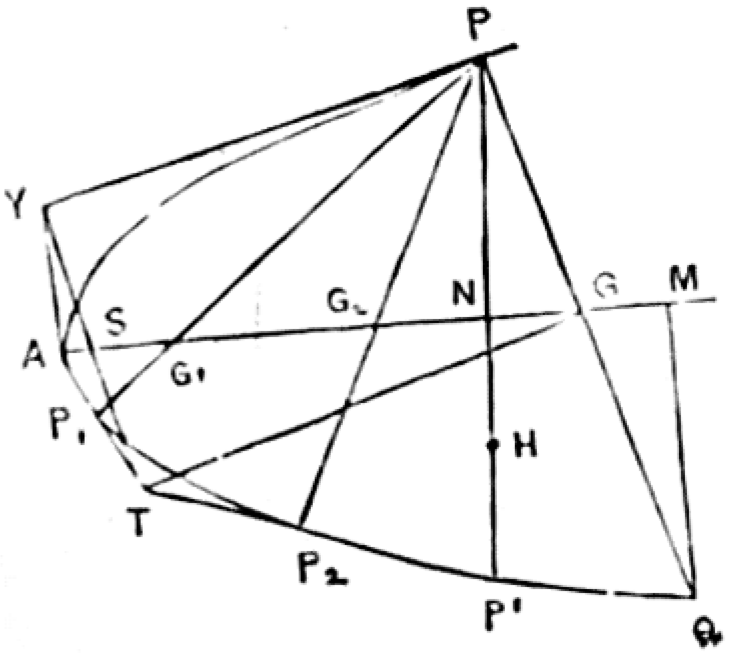
Hence the circle through P_1SP_2 passes through the circumcentre of $\Delta\ PP_1P_2$.

Note by A. A. Krishnaswami Aiyangar, B. A.

Let Y be the foot of the perpendicular from S on the tangent at P ; PP′ the double ordinate through P ; H the mid-point of NP′, and T the pole of the chord $P_1P.^2$

1. *The eight points A, Y, P, G, H, P_2, T, P_1 lie on a circle.*

For, the normal at 't' meets the curve again at the point '$-t-\dfrac{2}{t}$' which is evidently unaltered by changing $\dfrac{2}{t}$. Therefore the normals at t, $\dfrac{2}{t}$ meet on the curve at $\left(-t-\dfrac{2}{t}\right)$. Now the tangents at t, $\dfrac{2}{t}$ meet at $2a,\ a\left(t+\dfrac{2}{t}\right)$.



Thus if P_1, P_2, be t and $\frac{2}{t}$, P is $\left(-t-\frac{2}{t}\right)$ and the co-ordinates of T are $2a$, $a\left(t+\frac{2}{t}\right)$, so that if TL is drawn perpendicular to AS, TL$=\frac{1}{2}$ PN$=$NM or AY. That is the angle THP is a right angle and AL$=$2AS, proving that YS passes through T and YS$=$ST.

$\therefore$ PG$=$2YS$=$YT and PGTY is a rectangle.

Since angles YAG, PP_1T, TP_2P, and THP are right angles, the circle PGTY passes through A, P_1, P_2, H.

(2) *The centre of this circle lies on the circle SP_1P_2.*

$$P_1SP_2 = \text{double of the supplement of } P_1\hat{T}P_2.$$
$$= \text{double of } P_1\hat{P}P_2$$
$$= P_1\hat{X}P_2, \text{ where X is the centre of the circle } PP_1P_2.$$

Hence P_1, S, X, P_2 are concyclic.

Also, by elementary geometry, it is easily seen that SX cuts the circle PP_1P_2 in Y′ and Z′, which are the excentres of the triangle SP_1P_2.

Thus the circle PP_1P_2 passes through *seven* additional points, *viz.*: A, Y, G, H, T, and the excentres of the triangle SP_1P_2, opposite to P_1, P_2.

(3) Because AYGT is cyclic, therefore

$$AS.SP = AS.SG = ST^2 = SP_1.SP_2.$$

Question 560.

(N. SANKARA AIYAR M.A.):—Find the volume contained between the two ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ and } \frac{x^2}{b^2}+\frac{z^2}{c^2}+\frac{y^2}{a^2}=1.$$

Solution by T. P. Trivedi, M.A., LL.B., and T. P. Bhaskarasastri, B.A.

Suppose $a>b$; the area common to the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \text{ is given by}$$

$$4\left\{\frac{b}{a}\int_0^{\frac{ab}{\sqrt{a^2+b^2}}}\sqrt{a^2-x^2}\,dx+\frac{a}{b}\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{b}\sqrt{b^2-x^2}\,dx\right\}$$

[This is obvious on a little consideration; the ellipses intersect at points $\pm\frac{ab}{\sqrt{a^2+b^2}}, \pm\frac{ab}{\sqrt{a^2+b^2}}$ and the common area may be represented as the sum of two integrals.]

On simplifying, we find the common area to be

$$4\left\{\tfrac{1}{2}\,ab^{\sin-1}\frac{b}{\sqrt{a^2+b^2}}+\tfrac{1}{2}\,ab\left(\frac{\pi}{2}-\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right)\right\}$$

$$=4\,ab\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}.$$

If we take the sections of the ellipsoids by a plane parallel to the plane of xy and at a distance z from it, we get the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2}\text{ and }\frac{x^2}{b^2}+\frac{y^2}{a^2}=1-\frac{z^2}{c^2},$$

and the area common to these two is

$$4ab\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}$$

The volume common to tha two ellipsoids

$$=\int_{-c}^{c}4ab\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}dz.$$

$$=\frac{16\,abc}{3}\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}.$$

If $b=a$, the ellipsoids coincide and we get $\frac{4\pi a^2c}{3}$ for the common volume, as is otherwise obvious.

Question 566.

(S. **Krishnaswami Aiyangar**) :—Prove that

$$\sin^{-1}x=\frac{x}{(1-x^2)^{\frac{3}{2}}}\left\{1-s_2\frac{x^2}{1-x^2}+s_3\left(\frac{x^2}{1-x^2}\right)^2-\ldots\ldots\right\}$$

where s_r=the sum of the reciprocals of the first r odd numbers.

Solution by J. C. Swaminarayan, M.A., K. T. Sundara Rajan, B.A., R. J. Pocock B.A., B.Sc., F.R.A.S., and A. Narasinga Rao.

Now
$$\frac{\tan^{-2}z}{1+z^2}=\left(z-\frac{z^3}{3}+\frac{z^5}{5}-\ldots\ldots\right)\left(1-z^2+z^4-\ldots\ldots\right)$$

$$=z-(1+\tfrac{1}{3})z^3+(1+\tfrac{1}{3}+\tfrac{1}{5})z^5-\ldots\ldots$$

$$=z-s_2z^3+s_3z^5-\ldots\ldots$$

$$\therefore\quad \tan^{-1}z=z(1+z^2)\left\{1-s_2z^2+s_3z^4-\!-\ldots\ldots\right\}$$

$$\therefore\quad \sin^{-1}x=\tan^{-1}\left\{\frac{x^2}{\sqrt{1-x^2}}\right\}$$

$$=\frac{x}{(1-x^2)^{\frac{3}{2}}}\left\{1-s_2\frac{x^2}{1-x^2}+s_3\left(\frac{x}{1-x^2}\right)^2-\ldots\ldots\right\}$$

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Question 568.

(K. V. Anantanarayana Sastri, B. A.) :—Evaluate

$$\int_0^\infty \frac{\log x}{1+x^4}\,dx.$$

Solution (1) *by R. J. Pocock, J. C. Swaminarayan, T. P. Trivedi, M. A., LL. B. and* (2) *by J. M. Bose, M. A., B. Sc.*

(1) We have the integral

$$u=\int_0^\infty \frac{x^m dx}{1+x^n} \text{ (where } m<n)=\frac{\pi}{n} \operatorname{cosec} \frac{(m+1)\pi}{n}.$$

[Williamson p. **141**.]

Differentiating for m,

$$\int_0^\infty \frac{x^m \log x\, dx}{1+x^n}=-\frac{\pi^2}{n^2}\operatorname{cosec}\frac{(m+1)\pi}{n}\cot\frac{(m+1)\pi}{n}.$$

$$\text{If } m=0,\ \int_0^\infty \frac{\log x dx}{1+x^n}=-\frac{\pi^2}{n^2}\operatorname{cosec}\frac{\pi}{n}\cot\frac{\pi}{n}. \qquad \dots \qquad \dots \quad (1)$$

$$\text{If } n=4,\ \int_0^\infty \frac{\log x dx}{1+x^4}=-\frac{\pi^2}{16}\operatorname{cosec}\frac{\pi}{4}\cot\frac{\pi}{4}=-\frac{\pi^2}{8\sqrt{2}}.$$

If we differentiate r times for m, we get the integral

$$\int_0^\infty \frac{x^m (\log x)^r dx}{1+x^n}=\left(\frac{\pi}{n}\right)^{r+1}\frac{d^r}{d\theta^r}(\operatorname{cosec}\theta)$$

where $\theta=\frac{(m+1)\pi}{n}$.

(2) It will be perhaps more interesting if we work out the more general case

$$\int_0^\infty \frac{\log x}{1+x^n}dx$$

n being a positive integer.

On putting $x^n=v$ and $\frac{1}{n}=\alpha$ $(0<\alpha<1)$

$$\int_0^\infty \frac{\log x}{1+x^n}dx=\frac{1}{n^2}\int_0^\infty \frac{v^{\alpha-1}\log v}{1+v}dv.$$

Now assume

$$I=\int_0^\infty \frac{x^{\alpha-1}\log x}{1+x}dx,\ J=\int_0^\infty \frac{x^{\alpha-1}\log x}{1-x}dx.$$

Take a contour consisting of the part of the real axis from $-\infty$ to $+\infty$, two small semicircles Υ' and Υ of radii r' and r round the points $x=-1$, and the origin respectively, and a semicircle Γ of large radius, each on the upper half of the z—plane. Let the integral

$$\int_c \frac{z^{a-1}\log z}{1+z} dz$$

be taken round this contour. As the integrand has no singularities within

$$\int_c = \int_0^\infty + \int_\Gamma + \int_{-\infty}^{-1-r'} - \int_{\Upsilon'} + \int_{-1+r'}^{0} - \int_{\Upsilon} = 0 \qquad \dots \quad (1)$$

At all points on Γ, z is very large and the integrand is sensibly equal to $\frac{\log z}{z^{1-a}}$ which vanishes in the limit.

$$\therefore \qquad \int_\Gamma = 0.$$

At all points on Υ, z is small, and the integrand is sensibly equal to $z^{a-1}\log z$ so that $\int z^{a-1}\log z dz = \left[z^a\left(\frac{1}{a}-\frac{1}{a^2}\right)\right]_\Upsilon$ which vanishes when the radius of Υ is sufficiently diminished.

Hence $\int_\Upsilon = 0$; so that from (1)

$$\int_0^\infty dx \frac{x^{a-1}\log x}{1+x}\left(\int_{-\infty}^{-1-r'} + \int_{-1+r''}^{0}\right)\frac{z^{a-1}\log z\ dz}{1+z} = \Upsilon' \int_{\Upsilon'}.$$

Now $\int_{\Upsilon^1} = \pi i \times$ residue of $\frac{z^{a-1}\log z}{1+z}$ at $z=-1$

$$= -\pi^2 (-1)^{a-1}.$$

Again within the range $-\infty$ to 0, z is real and negative to that on putting $z=-x$.

$$\left(\int_{-\infty}^{-1-r'} + \int_{-1+r'}^{0}\right)\frac{z^{a-1}\log z}{1+z} dz$$

$$= \left(\int_\infty^{1+r'} + \int_{1-r'}^{0}\right)\frac{(-1)^{a-1}x^{a-1}(\log x + \pi i)}{1-x} dx$$

$$= (-1)^{a-1}\left(\int_\infty^{1+r'} + \int_{1-r^1}^{0}\right)\frac{x^{a-1}\log x}{1-x} dx$$

$$+ (-1)^{a-1}\pi i\left(\int_\infty^{1+r'} + \int_{1-r'}^{0}\right)\frac{x^{a-1}}{1-x} dx.$$

Now in the limit when r' is sufficiently diminished

$$\int_{\infty}^{1+r'} + \int_{1-r'}^{0} = \int_{0}^{\infty} \text{ and } \int_{0}^{\infty} \frac{x^{\alpha-1}}{1-x}\,dx = \pi \cot \alpha\pi.$$

(Williamson p. 141.)

So that finally

$$\int_{0}^{\infty} \frac{x^{\alpha-1}\log x}{1+x}\,dx + J(-1)^{\alpha-1}$$

$$+\pi^2 i(-1)^{\alpha-1}\cot \alpha\pi = -\pi^2(-1);^{\alpha-1}$$

or $$(-1)^{1-\alpha} I + J + \pi^2 i \cot \alpha\pi = -\pi^2.$$

Equating imaginary parts

$$I \sin\frac{\pi}{n} = -\pi^2 \cot\frac{\pi}{n}.$$

$$\therefore \qquad \int_{0}^{\infty} dx\,\frac{\log x}{1+x^n} = -\frac{\pi^2}{n^2}\,\frac{\cos\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}.$$

Putting $n=4$

$$\int_{0}^{\infty} \frac{\log x}{1+x^4}\,dx = -\frac{\pi^2}{8\sqrt{2}}.$$

Question 570.

(**Professor Sanjana**) :—Prove that when n and k are positive integers, the sums of the following series vanish :—

$$(a)\quad 1-\binom{n+2k}{2}\frac{1}{2n+2k-1}+\binom{n+2k}{4}\frac{1.3}{(2n+2k-1)(2n+2k-3)}$$

$$-\binom{n+2k}{6}\frac{1.3.5}{(2n+2k-1)(2n+2k-3)(2n+2k-5)}+\ldots\ldots\ldots$$

$$(b)\quad \binom{2n+2k+1}{1}-\binom{2n+2k+1}{3}\frac{2n+3}{2n+2k-1}$$

$$+\binom{2n+2k+1}{5}\frac{(2n+3)(2n+5)}{(2n+2k-1)(2n+2k-3)}-\ldots\ldots\ldots$$

where $\binom{x}{y}$ denotes xCy.

Solution by J. C. Swaminarayan, M.A.

(a) Now $\frac{d^{n+2k}}{dx^{n+2k}}(x^2)^{k-\frac{1}{2}}$ vanishes for all positive integral values of

n and k.

But $$\frac{d^{n+2k}}{dx^{n+2k}}=\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\left(k-\frac{5}{2}\right)\cdots\cdots\left(-n-k+\frac{1}{2}\right)\times$$
$$(x^2)^{-n-2k-\frac{1}{2}}(2x)^{n+2k}$$

where S stands for $$\left[1+\frac{(n+2k)(n+2k-1)}{\underline{|1}}\times\frac{1}{2^2}\times\frac{1}{(-n-k+\frac{1}{2})}\right.$$
$$+\frac{(n+2k)(n+2k-1)(n+2k-2)(n+2k-3)}{\underline{|2}}\times\frac{1}{2^4}$$
$$\times\frac{1}{(-n+k+\frac{1}{2})(-n-k+\frac{3}{2}}.$$

The series inside the brackets can easily be put in the form

$$1-\binom{n+2k}{2}\cdot\frac{1}{2n+2k-1}+\binom{n+2k}{4}\cdot\frac{1.3}{(2n+2k-1)(2n+2k-3)}$$
$$-\binom{n+2k}{6}\frac{1.3.5}{(2n+2k-1)(2n+2k-3)(2n+2k-5)}+\cdots\cdots$$

whence the result (*a*) follows at once.

(*b*) Dividing both the sides by $(2n+2k+1)$, we have to prove that

$$1-\frac{2n+2k}{\underline{|3}}(2n+3)+\frac{(2n+2k)(2n+2k-2)}{\underline{|5}}(2n+3)(2n+5)-\cdots\cdots=0,$$

or $$1-(n+k)\frac{2n+3}{3}+\frac{(n+k)(n+k-1)}{\underline{|2}}\times\frac{(2n+3)(2n+5)}{3.5}-\cdots\cdots=0.$$

Now $\frac{d^{n+k}}{dx^{n+k}}\left\{x^{-n-\frac{3}{2}}x^{n+k+\frac{1}{2}}\right\}$ or $\frac{d^{n+k}}{dx^{n+k}}\left\{x^{k-1}\right\}$ vanishes for all positive integral values of n and k.

The former is

$$x^{-n-\frac{3}{2}}(n+k+\tfrac{1}{2})(n+k-\tfrac{1}{2})(nk+-\tfrac{3}{2})\cdots\cdots\tfrac{3}{8}x^{\frac{1}{2}}$$
$$\times\left[1+{}_{n+k}C_1\frac{(-n-\frac{3}{2})}{(\frac{1}{2})}+{}_{n+k}C_2\frac{(-n-\frac{3}{2})(-n-\frac{5}{2})}{(\frac{3}{2})(\frac{5}{2})}+\cdots\cdots\right]=0.$$
$$\therefore\qquad 1-\binom{n+k}{1}\frac{2n+3}{3}+\binom{n+k}{2}\frac{(2n+3)(2n+5)}{3.5}-\cdots\cdots\Big]=0,$$

whence the given result (*b*) follows at once.

Question 571.

(S. RAMANUJAN) :—If $\frac{\pi\alpha}{2}=\log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$, shew that

$$\left(\frac{1^2+\alpha^2}{1-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)\cdots\cdots\cdots=e^{\frac{1}{2}\pi\alpha\beta}.$$

Solution by J. C. Swaminarayan, M.A., R. J. Pocock, B.A., B.Sc., F.R.A.S., A. Narasinga Rao and K. B. Madhava.

By expressing $1-\sin \frac{1}{2}\pi x$ in factors and taking the logarithmic differential co-efficient, we obtain

$$\frac{\pi}{2}\left\{\sec\frac{\pi}{2}x+\tan\frac{\pi}{2}x\right\}=\frac{2}{1-x}-\frac{2}{3+x}+\frac{2}{5-x}-\frac{2}{7+x}+\ldots\ldots$$

and $$\frac{\pi}{2}\left\{\sec\frac{\pi}{2}x-\tan\frac{\pi}{2}x\right\}=\frac{2}{1+x}-\frac{2}{3-x}+\frac{2}{5+x}-\frac{2}{7-x}+\ldots\ldots$$

$$\therefore \quad \frac{\pi x}{2}\sec\frac{\pi}{2}x=\frac{2x}{1-x^2}-\frac{3\cdot 2x}{3^2-x^2}+\frac{5\cdot 2x}{5^2-x^2}-\frac{7\cdot 2x}{7^2-x^2}+\ldots\ldots$$

$$\therefore \quad \int\frac{\pi x}{2}\sec\frac{\pi}{2}x\,dx=-\log(1-x^2)+3\log(3^2-x^2)-5\log(5^2-x^2)+\ldots$$

Changing x into iy, we get

$$\int\frac{\pi y}{2}\operatorname{sech}\frac{\pi y}{2}dy=\log(1+y^2)-3\log(3^2+y^2)+5\log(5^2+y^2)-\ldots$$

The given condition is equivalent to $\cosh\frac{\pi\alpha}{2}=\sec\frac{\pi\beta}{2}$.

$$\therefore \quad \tanh\frac{\pi\alpha}{2}\,d\alpha=\tan\frac{\pi\beta}{2}\,d\beta.$$

$$\therefore \quad d\alpha=\sec\frac{\pi\beta}{2}\,d\beta \text{ and } d\beta=\operatorname{sech}\frac{\pi\alpha}{2}\,d\alpha.$$

$$\therefore \quad \frac{\pi}{2}\alpha\beta=\int\frac{\pi}{2}(\beta d\alpha+\alpha d\beta)$$

$$=\int\frac{\pi}{2}\alpha\operatorname{sech}\frac{\pi\alpha}{2}\,d\alpha+\int\frac{\pi}{2}\beta\sec.\frac{\pi}{2}d\beta$$

$$=\log\frac{1+\alpha^2}{1-\beta^2}-3\log\frac{3^2+\alpha^2}{3^2-\beta^2}+5\log\frac{5^2+\alpha^2}{5^2-\beta^2}-\ldots$$

Hence $$\left(\frac{1+\alpha^2}{1-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\ldots\ldots\ldots=e^{\frac{1}{2}\pi\alpha\beta}.$$

Question 572.

(M. Bhimansena Rao) :—If $f(x)=x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}\ldots\ldots\ldots\ldots$, shew that

(i) $f(\sqrt{2}-1)=\frac{\pi^2}{16}-\frac{1}{4}\left\{\log(\sqrt{2}-1)\right\}^2$,

(ii) $f\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{12}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$,

(iii) $f(\sqrt{5}-2)=\frac{\pi^2}{24}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$.

Solution by J. C. Swaminarayan, M.A.

It is obvious that $f(0)=0$ and $f(1)=\frac{\pi^2}{8}$.

$$\text{(1) } f(\sqrt{2}-1)=\int_0^{\sqrt{2}-1}\frac{\tanh^{-1}x}{x}dx$$

$$=\left\{\tanh^{-1}x\log x\right\}_0^{\sqrt{2}-1}-\int_1^{\sqrt{2}-1}\frac{\log x}{1-x^2}dx-\int_0^1\frac{\log x}{1-x^2}dx$$

$$=\frac{\pi^2}{8}-\frac{1}{2}\left\{\log(\sqrt{2}-1)\right\}^2-\int_1^{\sqrt{2}-1}\frac{\log x}{1-x^2}dx.$$

If we put $x=\frac{1-z}{1+z}$, $\int_1^{\sqrt{2}-1}\frac{\log x}{1-x^2}dx$ becomes $\int_0^{\sqrt{2}-1}\frac{\tanh^{-1}z}{z}dz$ or $f(\sqrt{2}-1)$.

$\therefore\ 2f(\sqrt{2}-1)=\frac{\pi^2}{8}-\frac{1}{2}\left\{\log(\sqrt{2}-1)\right\}^2$, whence the result.

$$\text{(ii) }\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log(1+x)}{x}=\left\{\log(1+x)\log x\right\}_0^{\frac{\sqrt{5}-1}{2}}-\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log x}{1-x}dx$$

$$=-\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2+\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log(1-z)}{z}dz-\int_0^{\frac{\sqrt{5}-1}{2}}dz,\text{ where } x+1=\frac{1}{z}$$

$$=-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}+\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\log(1-x)}{x}dx+\frac{\pi^2}{6}.$$

$$\therefore\quad\int_0^{\frac{\sqrt{5}-1}{2}}\frac{1}{x}\log\frac{1+x}{1+x}dx=2\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\tanh^{-1}x}{x}dx$$

$$=\frac{\pi^2}{6}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

whence the result.

$$\text{(iii)}\ f(\sqrt{5}-2)=\int_0^{\sqrt{5}-2}\frac{\tanh^{-1}x}{x}\,dx=\Big\{\tanh^{-}x^{1}\log x\Big\}_0^{\sqrt{5}-2}$$

$$-\int_1^{\sqrt{5}-2}\frac{\log x}{1-x^2}\,dx-\int_0^{1}\frac{\log x}{1-x^2}\,dx.$$

$$=\frac{\pi^2}{8}-\frac{3}{2}\Big\{\log\frac{\sqrt{5}-1}{2}\Big\}^2-\int_1^{\sqrt{5}-2}\frac{\log x}{1-x^2}\,dx.$$

By putting $x=\frac{1-z}{1+z}$, $\int_1^{\sqrt{5}-2}\frac{\log x}{1-x^2}\,dx$ becomes $\int_0^{\frac{\sqrt{5}-1}{2}}\frac{\tanh^{-1}z}{z}\,dz$

or $f\Big(\frac{\sqrt{5}-1}{2}\Big)$.

$$\therefore f\Big(\sqrt{5}-2\Big)=\frac{\pi^2}{8}-\frac{3}{2}\Big\{\log\frac{\sqrt{5}-1}{2}\Big\}^2-\frac{\pi^2}{12}+\frac{3}{4}\Big\{\log\frac{\sqrt{5}-1}{2}\Big\}^2$$

$$=\frac{\pi^2}{24}-\frac{3}{4}\Big\{\log\frac{\sqrt{5}-1}{2}\Big\}^2.$$

Question 573.

(M. Bhimasena Rao) :—Shew that

$$\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\ldots\ldots=\frac{\pi^2}{12}-\frac{3}{2}\Big\{\log\frac{\sqrt{5}-1}{2}\Big\}^2$$

Solution by J. C. Swaminarayan, M.A.

Since

$$\frac{\tanh^{-1}x}{\sqrt{1+x^2}}=x-\frac{2}{3}x^3+\frac{2\cdot4}{3\cdot5}x^5-\frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7+\ldots\ldots$$

the given series is equal to

$$\int_0^{\frac{1}{2}}\frac{\tanh^{-1}x}{x\sqrt{1+x^2}}\,dx=\int_0^{\frac{1}{2}}\frac{2\tanh^{-1}\Big(\frac{\sqrt{1+x^2}-1}{x}\Big)}{x\sqrt{1+x^2}}\,dx$$

$$=\int_0^{\sqrt{5}-2}\frac{2\tanh^{-1}z}{z}\,dz,\ \text{where}\ \frac{\sqrt{1+x^2}-1}{x}=z.$$

$$=2\Big\{\frac{\pi^2}{24}-\frac{3}{4}\Big(\log\frac{\sqrt{5}-1}{2}\Big\}^2\Big\}\ \text{by (iii) of Q. 572.}$$

$$=\frac{\pi^2}{12}-\frac{3}{2}\Big(\log\frac{\sqrt{5}-1}{2}\Big)^2.$$

Question 578.

(N. P. PANDYA) :—Construct a triangle ABC having given the side BC, the distance between the circumcentre and the symmedian point, and also the distance between the incentre and the excentre.

Solution by V. V. S. Narayan.

The circumcircle bisects II_1, which is a diameter of the cyclic quadrilateral $BICI_1$.

Therefore take BC and describe a $\triangle BDC$ so that $BD = CD = \frac{1}{2}II_1$.

Then, the circle BDC will be the circumcircle and its centre O the circumcentre. Hence R is known and since OK is given, w, the Brocard angle of the triangle can be found geometrically from the well known relation

$$OK^2 = R^2(1 - 3\tan^2 w).$$

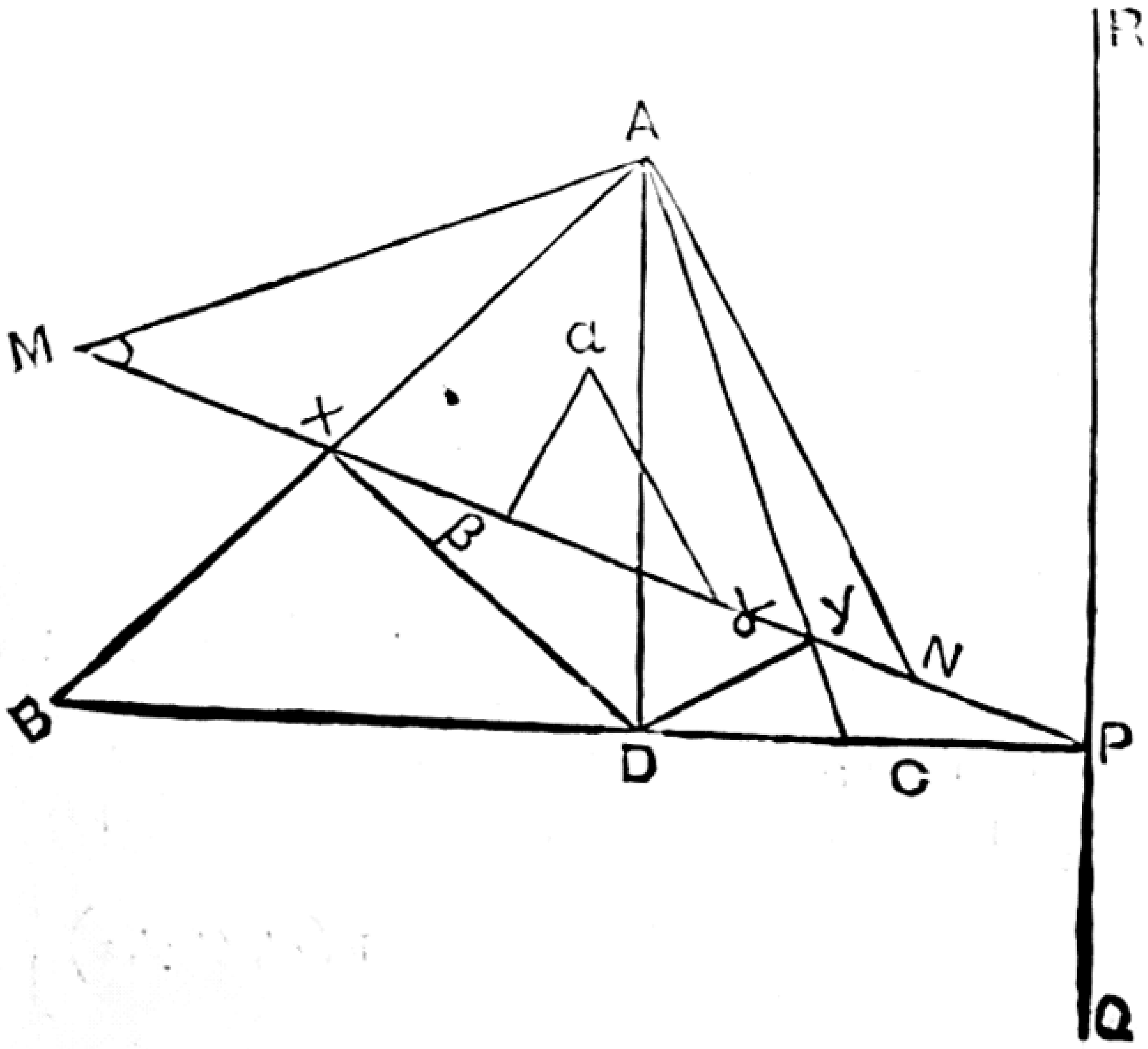
With the usual notation, we know that :—

$$\Omega OK = w, \text{ the Brocardangle}$$

and $$O\Omega K = \text{a right angle}.$$

$$\therefore \quad O\Omega = OK \cos w.$$

From this relation $O\Omega$ can be easily found geometrically, since OK is given and since w can be determined as given above. Then, draw $B\Omega$ so that $\Omega\hat{B}C = w$. With centre O and radius $O\Omega$ draw an arc to cut $B\Omega$ at Ω. Join $C\Omega$, draw CA making $\Omega\hat{C}A$; equal to w and cutting the circumcircle at A. Then ABC will be the required triangle.



QUESTIONS FOR SOLUTION

613. (P. V. SESHU AIYAR) :—Show that

$$\underset{z=0}{\mathrm{Lt}}\ \frac{\dfrac{\Pi(s)}{\Pi(s+z)}\left(\dfrac{x}{2}\right)^{z}-\dfrac{\Pi(n+s)}{\Pi(n+s-z)}\,\dfrac{x}{2}\Bigr)^{-z}}{z}$$

$$=2\log\frac{x}{2}-\psi(s)-\psi(n+s),$$

where $\psi(x)$ stands for $\frac{d}{dx}\log\Pi(x)$.

614. (M. T. NARANIENGAR) :—If ${}_1u_n=\left(\frac{1.3.5.\dots\dots2n-1}{2.4.6.\dots\dots2n}\right)^2$, prove that

$$u_n=4^{2n}u_{2n}-(4n-2)4^{2n-1}u_{2n-1}+\frac{(4n-3)(4n-4)}{1.2}4^{2n-2}u_{2n-2}-\dots\dots$$

615. (SELECTED) :—Find two numbers such that the sum of all the aliquot parts of either is equal to the other.

616. (S. NARAYANA AIYAR, M.A.) :—Establish the following extension of Maclaurin's theorem :—

$$\phi(x)+\frac{a-b}{b}\cdot\frac{x}{1}\phi'(x)+\frac{(a-b)(a-b-1)}{b.(b+1)}\,\frac{x^2}{1\cdot2}\phi''(x)+$$

$$\frac{(a-b)(a-b-1)(a-b-2)}{b.(b+1)(b+2)}\cdot\frac{x^3}{1\cdot2\cdot3}\phi'''(x)+\dots$$

$$=\phi(0)+\frac{a}{b}x\phi'(0)+\frac{a(a+1)}{b(b+1)}\cdot\frac{x^2}{1\cdot2}\phi''(0)+$$

$$\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\cdot\frac{x^3}{1\cdot2\cdot3}\phi'''(0)+\dots\dots$$

(*a*) When $a=b$, this reduces to Maclaurin's theorem.
(*b*) When $\phi(x)=e^x$, this reduces to Question 576 (I. M. J.)
(*c*) Examine the cases when $\phi(x)=\cos x$, $(1+x)^{-n}$, etc.

617. (S. NARAYANA AIYAR, M.A.) :—If $F(\alpha,\ \beta,\ \gamma,\ x)$ denotes the hyper-geometric series $1+\frac{\alpha.\beta}{\gamma.1}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot1\cdot2}x^2+\dots\dots$, shew that

$$\sum_{r=\infty}^{r=0}\left[\frac{\Gamma(b)\Gamma(\gamma)\Gamma(a-b+1)\Gamma(\alpha+r)\Gamma(\beta+r)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(b+r)\Gamma(\gamma+r)\Gamma(a-b-r+1)}\cdot\frac{x^r}{\Gamma(r+1)}\right.$$

$$\left.F\left(\alpha',\beta',\gamma',x\right)\right]$$

$$=1+\frac{a}{b}\cdot\frac{\alpha.\beta}{\gamma\cdot 1}x+\frac{a(a+1)\alpha(\alpha+1)\cdot\beta(\beta+1)}{b(b+1)\gamma(\gamma+1)1.2}x^2+\dots$$

where $\alpha'=\alpha+r$, $\beta'=\beta+r$, $\gamma'=\gamma+r$.

Examine the case when $\alpha=\gamma$.

618. (Professors K. J. SANJANA and T. P. TRIVEDI):—Establish the following identities, n being a positive integer and p any rational number $>2n$:

$$(1)\ 1-\frac{2n-1}{2^2\underline{|1}}\,\frac{p+2n+1}{2n-1}+\frac{(2n-2)(2n-3)}{2^4\,\underline{|2}}\,\frac{(p+2n+1)(p+2n-1)}{(2n-1)(2n-3)}$$

$$-\frac{(2n-3)(2n-4)(2n-5)}{2^6\,\underline{|3}}\,\frac{(p+2n+1)(p+2n-1)(p+2n-3)}{(2n-1)(2n-3)(2n-5)}+\dots$$

$$=\frac{(-1)^n}{2^{2n}}\,\frac{(p-1)(p-3)\dots\dots(p-2n+1)}{1.3.5\dots(2n-3)(2n-1)};$$

$$(2)\ 1-\frac{2n}{2^2\underline{|1}}\,\frac{p+2n}{2n-1}+\frac{2n(2n-3)}{2^4\underline{|2}}\,\frac{(p+2n)(p+2n-2)}{(2n-1)(2n-3)}$$

$$-\frac{2n(2n-4)(2n-5)}{2^6\,\underline{|3}}\,\frac{(p+2n)(p+2n-2)(p+2n-4)}{(2n-1)(2n-3)(2n-5)}+\dots$$

$$=\frac{(-1)^n}{2^{2n-1}}\,\frac{p(p-2)\dots(p-2n+)2}{1.3.5-(2n-3)(2n-1)};$$

$$(3)\ 1-\frac{(2n+1)^2-1^2}{\underline{|3}}\,\frac{p+2}{p+3}\,\frac{\{(2n+1)^2-1^2\}}{}$$

$$+\frac{\{(2n+1)^2-3^2\}}{\underline{|5}}\quad\frac{(p+2)(p+4)}{(p+3)(p+5)}-\dots$$

$$=\frac{(-1)^n}{2n+1}\,\frac{(p-1)(p-3)+\dots(p-2n+1)}{(p+3)(p-5)-\dots(p+2n+1)};$$

$$(4)\ 1-\frac{(2n+1)^2-1^2}{\underline{|2}}\,\frac{1}{p+3}$$

$$+\frac{\{(2n+1)^2-1^2\}\{2n+1)^2-3^2\}}{\underline{|4}}\quad\frac{1\cdot 3}{(p+3)(p+5)}-\dots\dots$$

$$=\frac{(p-1)(p-3)\dots(p-2n+1)}{(p+3)(p+5)\dots(p+2n+1)};$$

$$(5)\ 1-\frac{(2n)^2}{\underline{|2}}\,\frac{p+1}{p+2}+\frac{(2n)^2\{(2n)^2-2^2\}}{\underline{|4}}\,\frac{(p+1)(p+3)}{(p+2)(p+4)}-\dots\dots$$

$$=-(1)^n\frac{p(p-2)\dots(p-2n+2)}{(p+2)(p+4)\dots(p+2n)};$$

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SRINAGAR.

$$(6)\ 1-\frac{(2n)^2}{\underline{|2}}\,\frac{1}{p+2}+\frac{(2n)^2\{(2n)^2-2^2\}}{\underline{|4}}\,\frac{1\cdot3}{(p+2)(p+4)}-\dots$$

$$=\frac{p(p-2)\dots(p-2n+2)}{(p+2)(p+4)\dots(p+2n)}.$$

[Each of the series *terminates at the* $(n+1)$ *th* term.]

619. (S. KRISHNASWAMI AIYANGAR):—

If $$s_n=\frac{1}{(2n-1)^2(2n-3)^2}+\frac{1}{(2n-5)^2}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n-3)^2}\right\}$$

$$+\frac{1}{(2n-7)^2}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n-3)^2}+\frac{1}{(2n-5)^2}\right\}+\dots$$

shew that

$$\frac{s_2}{5^2}+\frac{s_3}{7^2}+\frac{s_4}{9^2}+\quad=\frac{\pi^6}{2^6\underline{|6}}.$$

620. (S. KRISHNASWAMI AIYANGAR):—Find the value of

$$1-\frac{3^5x^2}{2\,!}+\frac{5^5x^4}{4\,!}-\frac{7^5x^6}{6\,!}+\dots\dots\dots$$

621. (R. VYTHYNATHASWAMY):—If C is the chord of an arc λ of a closed curve, shew that $\int C ds$ taken over the contour is equal to λ times the perimeter of the locus of centroids of arcs of the curve equal to λ.

622. (R. VYTHYNATHASWAMY):—A moving quadrilateral has its four corners on a fixed circle. Shew that at any instant the sum of the angular velocities of the four pedal lines is equal to the sum of the angular velocities of the corners about the centre.

623. (M. BHIMASENA RAO):—Two triangles in perspective are such that the perpendiculars from the vertices of either, on the corresponding sides of the other, are concurrent. Shew that the two points of concurrence of these perpendiculars are in a line with the centre of perspective.

624. (A. N. RAGHAVACHAR):—Two parabolas of latera recta $4a$, $4b$ have a common vertex and their axes are inclined at an angle α. If they again intersect at an angle $(\alpha-\beta)$ and if the distance between their foci is c, prove that

$$\frac{c^2}{ab\cos\alpha}=\frac{9t^3s(3ts^2-4t+8s)}{4(t-2s)^2(2t-s)}.$$

625. (V. V. S. NARAYAN) :—Shew that the lines joining the vertices of a triangle to the corresponding vertices of its first Brocard triangle concur in a point Q whose trilinear co-ordinates are inversely proportional to those of the pole of $\Omega\,\Omega'$ with respect to the Brocard circle.

626.—(M. BHIMASENA RAO) :—If $f(x)$ denotes

$$\operatorname{sech}\frac{\pi x}{2}-\frac{1}{3}\operatorname{sech}\frac{3\pi x}{2}+\frac{1}{5}\operatorname{sech}\frac{5\pi x}{2}-\frac{1}{7}\operatorname{sech}\frac{7\pi x}{2}+\ldots$$

shew that

(i) $f(1)=\frac{\pi}{8}$

(ii) $f(\sqrt{3})=\frac{\pi}{24}$

(iii) $f(\sqrt{7})=\frac{1}{2}\left(\sin^{-1}\frac{3}{4}-\frac{\pi}{4}\right)$

(iv) $f(\sqrt{15})=\frac{1}{2}\left(\frac{\pi}{12}-\sin^{-1}\frac{1}{4}\right)$

(v) $f\left(\sqrt{\frac{3}{5}}\right)=\frac{1}{2}\left(\cos^{-1}\frac{1}{4}-\frac{\pi}{12}\right)$

(vi) $f(x)+f(x^{-1})=\frac{\pi}{4}$.

Are there any values of x other than zero and those given by (vi) for which $f(x)$ can be evaluated in finite terms ?

627. (K. V. ANANTANARAYANA SASTRI, B. A.) :—Four spheres of radii a, b, c, d, intersect at right angles. Shew that the volume of the tetrahedron formed by their centres is

$$\frac{1}{6}abcd(a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}.$$

628. (K. V. ANANTANARAYANA SASTRI, B. A.) :—Prove that the length of the first negative pedal of the loop of the Folium of Descartes $x^3+y^3-3axy=0$ is $6a-a(\pi-\sqrt{2}\log 1+\sqrt{2})$.

629. (S. Ramanujan) :—Prove that

$$\frac{\frac{1}{2}+\sum\limits_{n=1}^{n=\infty} e^{-\pi n^2 x}\cos(\pi n^2\sqrt{1-x^2})}{\sum\limits_{n=1}^{n=\infty} e^{-\pi n^2 x}\sin(\pi n^2\sqrt{1-x^2})}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{(1-x)}}\,;$$

and deduce the following:

(*a*)
$$\frac{\frac{1}{2}+\sum\limits_{n=1}^{n=\infty} e^{-\pi n^2}}{\frac{1}{2}+\sum\limits_{n=1}^{n=\infty} e^{-5\pi n^2}}=\sqrt{5\sqrt{5}-10}.$$

(*b*)
$$\sum_{n=1}^{n=\infty} e^{-\pi n^2}(\pi n^2-\tfrac{1}{4})=\tfrac{1}{8}.$$

List of Periodicals Received.

(From 16th November 1914 to 15th January 1915.)

1. American Journal of Mathematics, Vol. 36, No. 4, October 1914.
2. Annals of Mathematics, Vol. 16, No. 1, September 1914.
3. Astrophisical Journal, Vol. 40, Nos. 2 & 3, September and October 1914.
4. Bulletin of the American Mathematical Society, Vol. 21, Nos. 2 & 3, November and December 1914.
5. Bulletin des Sciences Mathematiques, Vol. 38, August, September and October 1914.
6. Educational Times, November and December 1914. (6 Copies.)
7. L'Intermediaire des Mathematiciens, Vol. 21, Nos. 7 & 8, July and August 1914.
8. Mathematical Gazette, Vol. 7, No. 113, October 1914, (4 copies)
9. Messenger of Mathematics, Vol. 44, Nos. 5 and 6, September and October 1914.
10. Philosophical Magazine, Vol. 28, Nos. 167 and 168, November and December 1914.
11. Popular Astronomy, Vol. 22, No. 9, November 1914, (3 copies).
12. Proceedings of the Edinburgh Mathematical Society, Vol. 32.
13. Proceedings of the London Mathematical Society, Vol. 13 & 14, Nos. 7 & 1, October, 1914.
14. Proceedings of the Royal Society of London, Vol. 91, Nos. 623 & 624, November and December 1914.
15. School Science and Mathematics, Vol. 14, Nos. 8 & 9, November and December 1914, (3 copies).
16. Transactions of the Cambridge Philosophical Society, Vol. 22, No. 5, October 1914.
17. Transactions of the Royal Society of London, Vol. 214, No. 518, 519 and 520.
18. American Mathematical Monthly, Vol. 21, No. 8, October 1914.

3. *Plane Trigonometry*—by H. S. Carslaw (corrected edition), Macmillan & Co., London, 1915, 4*s.* 6*d.*;

4. *The Laws of Algebra*—by A. G. Craknell, Univ. Tut. Press, London, 1915, 1*s.*

5. *Junior Algebra*—by A. G. Cracknell & A. Barraclough, Univ. Tut. Press, London, 1915, 2*s.*

6. According to Art. VI of the Bye-laws the audited Balance Sheet for the year 1914, and the Budget for the current year are circulated. In doing so, the Committee cannot but notice with a sense of apprehension the *decrease* in the receipt of subscriptions. It is confidently expected that members will try their utmost to pay off arrears and current subscriptions.

POONA,
31st March 1915.

D. D. KAPADIA,
Hony. Joint Secretary.

Balance Sheet for 1914.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Opening Balance from 1913 ...	970	1	9	Books and Journals ...	287	4	9
Subscriptions from Members ...	629	0	0	Library ...	309	4	0
" for Journal.	122	13	0	Journal Printing ...	433	1	6
Miscellaneous ...	0	14	0	Ordinary Working Expenses ...	184	8	4
				Closing Balance ...	508	10	2
	1,722	12	9		1,722	12	9

MADRAS,
12*th February* 1915.

(Sd.) S. NARAYANA AIYAR,
Ag. Hon. Treasurer.

I have examined the Treasurer's books and vouchers and the monthly statements submitted by the Secretary, Assistant Secretary and Assistant Librarian and declare the above accounts to be correct.

MADRAS,
12*th February* 1915.

(Sd.) S. KRISHNASWAMI AIYAR,
Auditor.

Budget for 1915.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Opening Balance ...	508	10	2	Books and Journals ...	350	0	0
Subscriptions from Members ...	2,100	0	0	Library ...	350	0	0
Sale of Journals ...	150	0	0	Printing Journal ...	550	0	0
				Ordinary Working Expenses ...	250	0	0
				Closing Balance ...	1,258	10	2
	2,758	10	2		2,758	10	2

MADRAS,
9*th February* 1915.

(Sd.) S. NARAYANA AIYAR,
Ag. Hon. Treasurer.

So-called Cases of Failure in the Solution of Linear Differential Equations.

By Eric H. Neville.

(*Continued from page* 6.)

We turn now to the question of the generality of the solution of equation (2) obtainable by means of equation (3), that is, to the problem of the construction of integrals of the equation

$$G(y)=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

In passing we remark that it is shewn by the last example that integrals of equation (2) deduced from different functions satisfying the conditions of equation (3) may differ, so that a number of independent functions satisfying equations of the form of (3) may lead to an equivalent number of independent solutions of (2), or may lead to any smaller number other than zero.

Again, any arbitrary constants occurring in the function $F(x, p_1, p_2, \ldots p_r)$ may be regarded as arbitrary functions of any or all of the parameters $p_1, p_2 \ldots p_r$ without affecting the relation between the equations (2), (3), the only conditions which may have to be imposed being negative conditions, preventing them or certain combinations of them from becoming zero or infinite for the special values $a_1, a_2 \ldots a_r$ of the parameters; it is easy to prove, from the way in which the arbitrary constants must enter into the function F, that the derived solution of equation (2) contains the same number of arbitrary constants as the given function.

Considerations of a different kind appear when we relate the solution of equation (17), or in other words the discovery of terms in the complementary function of equation (2), to the given result (3).

The first conclusion from the equation (3) in this connection is obvious: if for any set of values $b_{t1}, b_{t2}, \ldots b_{tr}$ of $p_1, p_2, \ldots p_r$ the function $h(p_1, p_2, \ldots p_r)$ is zero, the function $F(x, b_{t1}, b_{t2}, \ldots b_{tr})$ is a solution of equation (17). We may call such a set a zero-set of the function h. If there is only one parameter p, the values of p for which $h(p)$ vanishes are fixed numbers, the roots of the equation $h(p)=0$, but if there are more parameters than one, we can, as a rule, fix arbitrarily all but one of the parameters and determine the remaining one by the condition that the function vanishes; this remaining parameter is a function of the parameters which we consider arbitrary, and if one of the latter varies, the dependent parameter varies also. In fact, if there are more parameters than one, the values forming a zero-set are the values $c_1, c_2, \ldots c_s$. Then although a variation of these parameters will usually change the positions of the zeros of the function $h_u(p)$, causing coalescence and separation, it will not as a rule cause any change in the total order of this function, and when the order of the function is unchanged so also is its value for the determination of solutions of the differential equation $G_u(y)=0$.

For example, although the particular solutions e^{lx}, e^{mx} of the equation

$$(D-l)(D-m)y=0$$

which are suggested by the result

$$(D-l)(D-m)\epsilon^{px}=\{(p-l)(p-m)\}\,e^{px}$$

coalesce when l and m are equal, the corresponding identity

$$(D-l)^2e^{px}=(p-l)^2\epsilon^{px}$$

still has a quadratic function of p for a factor in the right hand side, and is still effective for suggesting two solutions of the equation

$$(\mathrm{D}-l)^2\,y=0,$$

namely ϵ^{lx} and xe^{lx}.

But it is to be observed that neither in this particular case nor in general is it necessary or even useful to have acquaintance with the solution of a more general equation under which a proposed equation is included. Unless the general equation happens to be such as to suggest to us a function $F(x, p)$ leading to a function $h(p)$ radically different from any such function already known, it is unlikely that by attacking the general equation we shall make any advance in solving the particular one.

The best illustration of this part of the theory is to be found in the method devised by Frobenius for constructing regular integrals of homogeneous linear equations. A function $F(x, p)$ is formed such that

$$G\{F(x, p)\}=h\,(p)\,x^p,$$

this function not becoming infinite identically for any finite value of p and $h(p)$ being a polynomial in p; the function $F(x, p)$ is constructed as a power series in x whose co-efficients are rational functions of p, and it is not always convenient to prevent these co-efficients from having a common factor in the numerator, though they must not have one in the denominator; for some of the zeros of $h(p)$ the function $F(x, p)$ may vanish identically, and it has been proved by Fuchs and by Frobenius (for references, a proof, and examples, see Forsyth's *Theory of Differential Equations*, Vol. IV) that the remaining zeros, which are the efficient zeros of the polynomial, are always sufficiently numerous to lead to all the regular integrals of the equation $G(y)=0$.

The precise result is that if the efficient zeros are divided into groups so that the real part of each zero differs from the real part of every zero in its own group by an integer and from the real part of every zero in every other group by an amount which is not an exact integer, and if the members of each group are so arranged that the real part of each is greater than the real part of the zero which follows it in its own group, then multiplicity q_{kl} of b_{kl}, the lth member of the kth group, is greater than the multiplicity $q_{k,\,l-1}$ of the zero $b_{k,\,l-1}$ which precedes it in its own group, and if we denote the value of $\partial^m F/\partial p^m$ when p has the value b_{kl} by $F_{kl}(m)$, the set of functions obtained by forming from each zero b_{kl} the $q_{kl}-q_{k,\,l-1}$ functions

$$F_{kl}(q_{k,\,l-1}),\ F_{kl}\,(q_{k,\,l-1}+1),\ F_{kl}(q_{k,\,l-1}+2),\ldots\ldots F_{kl}\,(q_{kl}-1)$$

is a complete set of linearly independent regular integrals of the equation $G(y)=0$ (the functions associated with the first member b_{k1} of the group are the values of F itself and of its first $q_{kl}-1$ derivatives, that is, are

$$F_{k1}(0),\ F_{k1}(1),\ldots\ldots F_{k1}\,(q_{k1}-1).$$

It will be observed that the number of functions thus obtained from the first l members of a group is equal to the multiplicity of the lth member itself, and that from this lth member alone we can obtain this number of integrals, namely F_{kl} (0), F_{kl} (1),... F_{kl} $(q_{kl}-1)$, but examples can be found to shew that the first $q_{k,\,l-1}$ members of this last set of functions are not necessarily linearly independent and indeed that some or all of them may vanish identically. The total number of integrals obtainable is the sum of the multiplicities of the last members of the various groups, and this is equal to or less than the order of the equation according as it is all or only some of the integrals of the equation which are regular.

In conclusion, we must return to the construction of integrals of equation (17) on the basis of an equation of the form (3) involving more than one parameter. There are two possibilities, illustrated by the examples already given, which might be thought distinct: the vanishing of the function $h(p_1, p_2,\ldots p_r)$ may be due solely to the taking of a particular value b_{tk} by one of the parameters p_k, independently of the other parameters, or it may be due to the satisfying of a relation really determining one parameter in terms of the others. But there is nothing vital in this distinction; in either case $r-1$ of the parameters remain arbitrary, and the function $F(x, p_1, p_2,\ldots p_r)$ is a function from which in general not all the arbitrary elements will disappear in virtue of the one equation $h(p_1, p_2,\ldots p_r)=0$. It is natural to regard as the general case that in which, after substituting in $F(x, p_1, p_2, \dots p_r)$ for one of the parameters in terms of the others in accordance with the equation $h(p_1, p_2, \dots pr)=0$, the function remains a function of x and of $r-1$ parameters; then it is a solution of the differential equation (17) involving $r-1$ constants, so that $r-1$ cannot be greater than n, the order of the equation, and if $r-1$ is equal to n, the solution is general. But it is equally conceivable for $F(x, p_., p_2, \dots p_r)$ to be of the form

$$H(x)+h(p_1, p_2 \dots p_r)\ K(x, p_1, p_2, \dots p_r)$$

in which case equation (3) assists us to only a particular solution of equation (17); any intermediate case is possible, and in fact the real difference between various functions $h(p_1, p_2, \dots p_r)$ from the point of view of utility in this connection resides in the generality remaining in $F(x, b_{t1}, b_{t2}, \dots b_{tr})$ where $b_{t1}, b_{t2} \dots b_{tr}$ is a zero-set of the function $h(p_1, p_2, \dots p_r)$. And what it is important to observe is that if a zero-set varies continuously it may yield always the same integral of the equation (17), or it may yield any number of integrals which are analytically distinguishable, but it cannot yield more than $r-2$ functions which are linearly independent solutions of the differential equation.

Multiplicity of a zero-set is the last point to which we have to refer. If the function h together with one of its derivatives of order m and all those derivatives of lower orders which are subsidiary to this particular derivative vanishes for the values $b_{t1}. b_{t2} \dots b_{tr}$ of the parameters, then the function F and all its corresponding derivatives including that of the mth order, assume for this set of values of the parameters forms which are integrals of equation (17).

Again we may illustrate from equation (10) how the fact of having more than one parameter may be utilised. To obtain more than one solution of the equation

$$(D^2-2D+2)y=0$$

from the one-parameter results (14), (15), (16) we have to introduce complex numbers, and then the factors of $F(x, p)$ exchange parts in a curious fashion. The roots of $h(p)$ are $1, i, -i$, all simple, the function $F(x, 1)$ is real, but the functions $F(x, i)$, $F(x, -i)$ are complex, and

$$(1+i)\ F(x, 1)+iF(x, i)+F(x, -i)=0.$$

If we take as our basis the two-parameter results (11), (12), (13) we have a zero-set of $h(p, q)$ which varies continuously, and $-2(p-1)\, qe^{(p+iq)x}$ is an integral for all values of p, q satisfying the equation $h(p,q)=0$. The only pairs of real values of p, q such that $h(p, q)=0$ are $p=1$, $q=\pm 1$; these values cause the integral just found to vanish identically, so that a certain trivial amount of justification has to be made of the statement that the finite functions $e^{(1+i)x}$, $e^{(1-i)x}$ are in fact integrals, and even then the functions are complex. Trouble is avoided by remarking that the sets $(1, 1)$ and $(1, -1)$ are double zero-sets of $h(p, q)$ with respect to both the parameters p, q, so that four particular integrals (a number equal to the degree of $h(p, q)$ in the two parameters) are given as the values of $\partial F/\partial p$ and $\partial F/\partial q$ for the two sets. The two values of $\partial F/\partial p$ are $\mp 2e^x \cos x$ and are not linearly independent, and the two values of $\partial F/\partial q$ are $\mp 2e^x \sin x$, and these though not linearly independent of each other are linearly independent of those already found. The equation being of order two, its complete solution is now in our hands, but we may notice that the values of $\partial^2 F/\partial p\, \partial q$ for the two zero-sets are also particular integrals; as a matter of fact they coincide, both of them being $-2e^x \cos x$.

Some Infinite Determinants.

By S. Narayana Aiyar, M. A.

Lemma.

If

$$\begin{vmatrix} a_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & \cdots & \cdots & \cdots & \cdots & a_1 & 1 \\ a_n & a_{n-1} & \cdots & \cdots & \cdots & \cdots & a_2 & a_1 \end{vmatrix} = (-1)^n A_n,$$

then

$$\begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_2 & A_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ A_3 & A_2 & A_1 & 1 & 0 & \cdots & 0 & 0 \\ A_4 & A_3 & A_2 & A_1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1} & A_{n-2} & \cdots & \cdots & \cdots & \cdots & A_1 & 1 \\ A_n & A_{n-1} & \cdots & \cdots & \cdots & \cdots & A_2 & A_1 \end{vmatrix} = (-1)^n a_n. \quad \ldots \quad (1).$$

Demonstration. By giving to n successive integral values from unity upwards and by developing the determinants thus obtained from the first determinant, we have the relations

$$A_1 + a_1 = 0$$

$$A_2 + A_1 a_1 + a_2 = 0.$$

$$A_3 + A_2 a_1 + A_1 a_2 + a_3 = 0.$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$A_n + A_{n-1} . a_1 + A_{n-2} . a_2 + \ldots A_2 . a_{n-2} + A_1 . a_{n-1} + a_n = 0. \quad \ldots \quad (2)$$

These relations are symmetrical in A and a. If the second determinant should be true the same relations should hold. Hence the *Lemma.*

Other demonstrations may be given but the above seems to be the simplest.

Prop. I. The infinite determinant

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 & \dots & \text{to } \infty \\ a_1 & 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ a_2 & a_1 & 1 & 0 & 0 & 0 & \dots & \dots \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \dots & \dots \\ a_4 & a_3 & a_2 & a_1 & 1 & 0 & \dots & \dots \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \text{to } \infty & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

is equal to

$$\frac{1}{1+a_1 x+a_2 x^2+a_3 x^3+a_4 x^4+a_5 x^5+\dots\dots} \qquad \dots (3)$$

Demonstration. It is easily seen that the minor of x^n in the above determinant is $(-1)^n A_n$. Hence expanding the determinant in terms of the constituents of the first row we have it equal to

$$1+A_1 x+A_2x^2+A_3x^3+A_4x^4+\dots \qquad \dots \qquad \dots (a)$$

Taking the same determinant with capital A's substituted for the small a's, the resulting determinant will, by the *Lemma* above, be equal to

$$1+a_1 x+a_2 x^2+a_3 x^3+a_4 x^4+\dots\dots\dots \qquad \dots \qquad \dots (b)$$

Multiplying the two expressions (a) and (b) and arranging the result in ascending powers of x we find, by the results (2) given in the above *Lemma*, the co-efficient of every power of x is zero. Hence the two expressions $1+A_1 x+A_2 x^2+A_3 x^3+\dots\dots$and $1+a_1x+a_2x^2+a_3x^3+\dots$ should be reciprocals when x is less than unity.

Hence the proposition.

Example 1.

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & \dots \\ \alpha & 1 & 0 & 0 & 0 & \dots \\ \beta & \alpha & 1 & 0 & 0 & \dots \\ 0 & \beta & \alpha & 1 & 0 & \dots \\ 0 & 0 & \beta & \alpha & 1 & \dots \\ 0 & 0 & 0 & \beta & \alpha & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \frac{1}{1+\alpha x+\beta x^2}.$$

If a and b are the roots of the equation $1+\alpha x+\beta x^2=0$, expressing

$$\frac{1}{1+\alpha x+\beta x^2} \text{ as equal to } \frac{1}{x(a-b)}\left\{\frac{1}{1-ax}-\frac{1}{1-bx}\right\}$$

and expanding separately each term in powers of x the co-efficient of x^n is $\frac{a^{n+1}-b^{n+1}}{a-b}$. The minor of x^n in the above determinant is

$$\begin{vmatrix} \alpha & 1 & 0 & 0 & 0 & \cdots \\ \beta & \alpha & 1 & 0 & 0 & \cdots \\ 0 & \beta & \alpha & 1 & 0 & \cdots \\ 0 & 0 & \beta & \alpha & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (4)$$

a determinant of the nth order.

Hence the value of this determinant of the n^{th} order is $\frac{a^{n+1}-b^{n+1}}{a-b}$ where a and b are the roots of the equation $1+\alpha x+\beta x^2=0$.

The same value in another form is given in Burnside and Panton, Vol. II, page 63.

Ex. 2. Consider the determinant

$$\begin{vmatrix} -2\cos\theta & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2\cos\theta & 1 & 0 & 0 & \cdots \\ 0 & 1 & -2\cos\theta & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (5)$$

This is of the n^{th} order and is the co-efficient of x^n in the expansion of $\frac{1}{1-2x\cos\theta+x^2}$ in powers of x and is therefore equal to $\frac{\sin(n+1)\theta}{\sin\theta}$.

Ex. 3. Let $1+p_1x+p_2x^2+p_3x^3\ldots+p_nx^n$ be equal to

$$(1-x\alpha)(1-x\beta)(1-x\gamma)\ldots\ldots(1-x\nu),$$

then the symmetric function

$$\sum\frac{\alpha^{n+r-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\ldots(\alpha-\nu)}$$

is equal to the determinant of the rth order

$$\begin{vmatrix} p_1 & 1 & 0 & 0 & \cdots & 0 \\ p_2 & p_1 & 1 & 0 & \cdots & 0 \\ p_3 & p_2 & p_1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_r & p_{r-1} & \cdots & \cdots & \cdots & p_1 \end{vmatrix} \qquad (6)$$

For, expressing $\frac{1}{1+p_1x+p_2x^2+\ldots p_nx^n}$ as the sum of n partial fractions and equating the co-efficient of x^n in the expansion of these fractions with the minor of x^n in the corresponding determinant, we get the above result.

Ex. 4.
$$e^{-x}=\frac{1}{1+x+\frac{x^2}{1\cdot2}+\frac{x^3}{1\cdot2\cdot3}+\frac{x^4}{1\cdot2\cdot3\cdot4}+\dots}$$

and is therefore equal to the infinite determinant

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2!} & 1 & 1 & 0 & 0 & \dots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & 0 & \dots \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & \dots \end{vmatrix} \qquad (7)$$

These examples can be indefinitely multiplied by giving particular values to $a_1, a_2 \dots\dots\dots$

Prop. II. The fraction

$$\frac{1+b_1x+b_2x^2+b_3x^3+\dots\dots\dots\text{to } \infty}{1+a_1x+a_2x^2+a_3x^3+\dots\dots\dots\text{to } \infty}$$

is equal to the infinite determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \text{to } \infty \\ 0 & 1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & \dots \\ b_1 & a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ b_2 & a_2 & a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_3 & a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \dots \\ b_4 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & 0 & \dots \\ \text{to } \infty & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \qquad (8)$$

Demonstration. By expanding the determinant in terms of the constituents of the first column we get the determinant given in *Prop. I* multiplied by $1+b_1x+b_2x^2+b_3x^3+\dots$ Hence the above is equal to the fraction

$$\frac{1+b_1x+b_2x^2+b_3x^3+b_4x^4+\dots}{1+a_1x+a_2x^2+a_3x^3+\dots\dots\dots}$$

Cor. By putting $a_1=a_2=a_3=\dots\dots=0$, we have

$$1+b_1x+b_2x^2+b_3x^3+b_4x^4+\dots\text{to } \infty$$

equal to the infinite determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \text{to } \infty \\ 0 & 1 & x & x^2 & x^3 & \cdots \\ b_1 & 0 & 1 & 0 & 0 & \cdots \\ b_2 & 0 & 0 & 1 & 0 & \cdots \\ \text{to } \infty & & \cdots & \cdots & & \cdots \end{vmatrix} \qquad (9).$$

Example 5. The determinant given in *Prop. II* is equal to the infinite determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & x & x^2 & x^3 & x^4 & x^5 & \cdots \\ A_1 & B_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ A_2 & B_2 & B_1 & 1 & 0 & 0 & 0 & \cdots \\ A_3 & B_3 & B_2 & B_1 & 1 & 0 & 0 & \cdots \\ A_4 & B_4 & B_3 & B_2 & B_1 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (10)$$

where

$$\begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ a_r & a_{r-1} & & \cdots & \cdots & a_1 \end{vmatrix} = (-1)^r A,$$

and

$$\begin{vmatrix} b_1 & 1 & 0 & \cdots & 0 \\ b_2 & b_1 & 1 & \cdots & 0 \\ b_3 & b_2 & b_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_r & b_{r-1} & \cdots & \cdots & b_1 \end{vmatrix} = (-1)^r B_r.$$

This can be proved from the relation

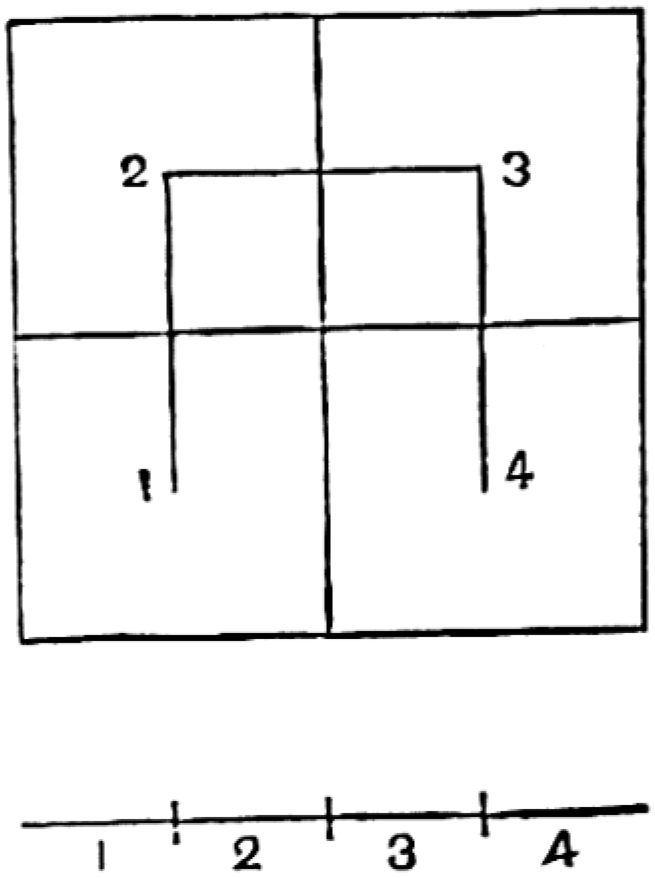
$$\frac{1}{1+a_1x+a_2x^2+\ldots} = 1+A_1x+A_2x^2+A_3x^3+\ldots,$$

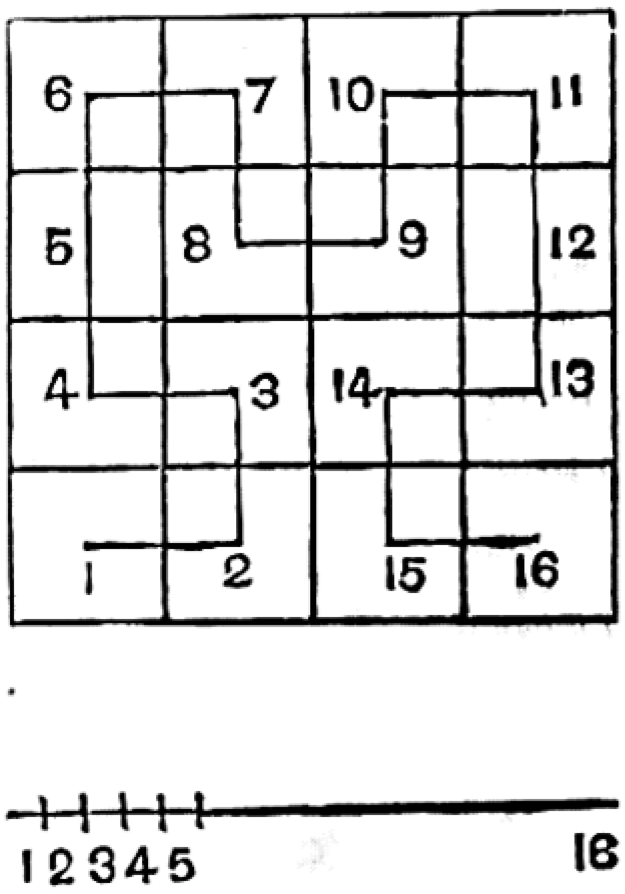
SHORT NOTES.

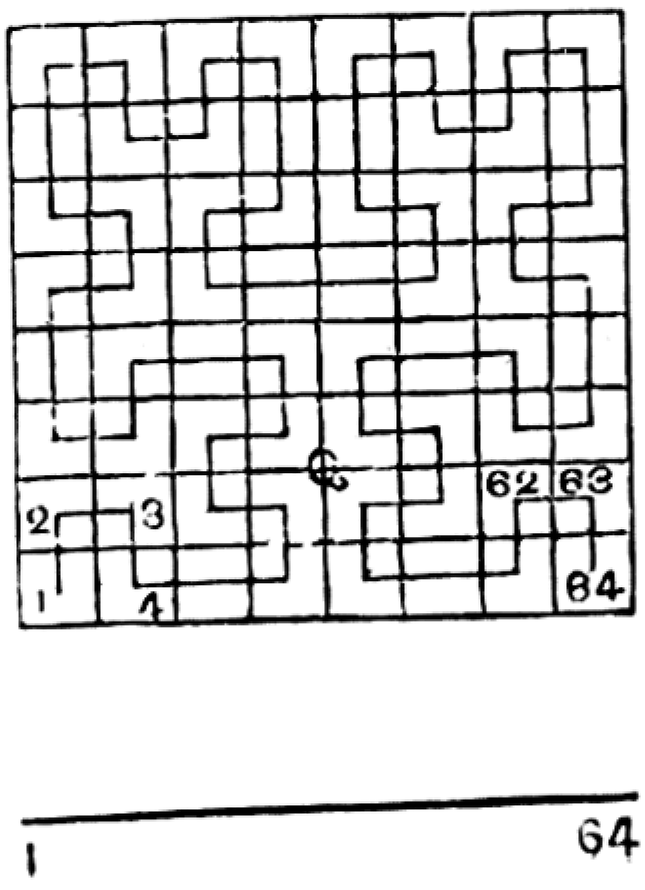
Peano's 'Space filling Curves.'

The notion of a curve, though it may appear simple, is in reality very complex, and often, when the co-ordinates of any point on it are given as functions of a parameter t, a rigorous analytical reasoning leads to results not in conformity with geometric intuition. Peano, an Italian Mathematician, was the first to draw the attention of mathematicians to an extremely curious circumstance; *viz.*, that it is possible to form continuous functions $f(t)$ and $\phi(t)$ such that by varying the parameter t, the point (x, y), given by $x=f(t)$, $y=\phi(t)$, may be made to coincide with all the points inside a square. We thus see that we can obtain, in a way, a 'curve' which fills entirely the area of a square.

The following, which is a modification given by Hilbert of Peano's construction, gives an extremely simple example of a curve possessing the preceding property.







Take a segment of a straight line of unit length and a square of side unity. Then divide the segment of the straight line into four equal parts and the square into four equal squares ; and number the segment of the straight line and the partial squares as in fig. (1). Divide similarly each of the segments of the straight line into four equal parts and each of the partial squares into four others; and number the sixteen squares in the order indicated in fig. (2) so that adjacent segments on the straight line correspond to adjacent squares and each square has a side in common with the succeeding square. Continuing this division of each of the segments of the straight line and each of the partial squares into four equal parts, we obtain 64 squares numbered as in fig. (3) so that as before adjacent segments of the straight line correspond to adjacent squares and each square has a side in common with the succeeding square. We thus see that to each segment upon the straight line (0,1) corresponds a definite square.

We may continue indefinitely this operation. It is then easy to see how to each point of the straight line (0,1) corresponds a point of the initial square. Let p be a point of the straight line. If it is not a point of division, then in the different stages of sub-division there is always a segment containing the point p ; and all these different segments, one from each stage, lie one inside the other and tend towards the point p; to each of the segments corresponds a square and all these squares also lie one within the other ; they diminish indefinitely and tend towards a limiting point, say P, which we take as the point of the square corresponding to the point p. If on the other hand p is a point of division, it determines two series of segments one lying on its right and the other on its left and each having p as the common end-point; corresponding to these two series of segments we have two series of squares. but since the square in one series at any stage is always adjacent to the square in the other series at the same stage, both series define the same point P, which is the common boundary point. Thus again we have corresponding to p a single point P.

Next this correspondence is also continuous. For an interval of length not greater than d (where $4^{-(k+1)} \leqq d < 4^{-k}$) with the point p as centre will overlap, at most, into two adjacent intervals at the k^{th} stage ; therefore two, at most, of the corresponding squares and these adjacent to one another will suffice to contain a correspondent for each interval point of the interval d ; and since the diagonal length of two such squares is at most $\frac{\sqrt{5}}{2^k}$, a circle C with centre P and radius

$r>\frac{\sqrt{5}}{2^k}$, will certainly enclose a correspondent for each interval point of d. Now whatever circle C be described having P as centre, we can always determine k so that $\frac{\sqrt{5}}{2^k}$ is less than its radius, and if we take the length of d to lie between $4^{-(k+1)}$ and 4^{-k}, it follows that the correspondent of each and every internal point of d is inside the circle C. Therefore the correspondence is a continuous one.

Also this correspondence is such that corresponding to a point p there is one and only one point P. We may therefore say that, if (x, y) denote the co-ordinates of P, we have $x=f(t), y=\phi(t)$ $(0\leq t\leq 1)$, t being the distance of p from the origin, where f and ϕ are continuous single-valued functions of t. We thus have a '*continuous line*' filling up completely a unit square.

It should, however, be noted that this is not a case of *one-to-one* correspondence. For, though for every point p of the straight line (0,1) we have one and only one point P of the square, yet corresponding to a point P of the square, we may have one, two, or four points of the straight line. If P be a point of the square which does not lie on a dividing line at any stage, then there is one and only one point p of the straight line corresponding to it. If P lies on a dividing line but not at a corner of a square, it will have two different correspondents on the straight line one for each series of squares lying one inside the other and having P as common boundary point. Lastly, if P is at a corner of a square where the dividing lines intersect as the point q in fig (3), it will have four distinct correspondents.

The functions $f(t)$ and $\phi(t)$ thus obtained being continuous functions of t, they also offer a very simple example of *continuous functions not having derivatives*. For, if these functions have derivatives, there will be an arc for the curve considered which is clearly absurd since the curve fills the whole square.

Now a question may be raised whether such a line as is described above can be called a 'curve.' The answer to this will depend on the definition of a *curve*. Jordan originally defined a curve as a 'continuous line" *i.e.*, as "a plane set of points which can be brought into continuous correspondence with the points of a closed segment of a straight line, and raised a query whether it was possible for such a curve to fill up completely a space. In answer to this query Peano gave a construction similar to the above and showed that a curve thus defined may fill up entirely a space; and hence the expression "space filling curve" has crept into mathematical literature. As this curious circumstance pointed out by Peano does violence to our common concept of a curve, a further restriction has since been introduced into the definition of a curve, *viz.*, that the correspondence should be *one-to-one*; *i.e.*, in analytical definition by $x=f(t)$ and $y=\phi(t)$, f and ϕ must be continuous single-valued functions of t, within given limits, which *do not assume the same pair of values for any two different values of* t within the given limits.

For further information, the reader is referred to Young's *Theory of Sets of Points* and Jordan's *Cours d' Analyse.*

P. V. Seshu Aiyar.

The Generalized Cassinian.

1. The locus of a point which moves in such a way that the product of tangents from it to two fixed circles is constant may be called the *generalized* Cassinian.

2. *The equation of the curve.*

Let the fixed circles be

$$S\equiv x^2+y^2+2gx+k=0,$$
$$S'\equiv x^2+y^2+2g'x+k=0,$$

referred to the line of centres and the radical axis as axes of reference. Then, if (x, y) be any point P on the locus, the tangents t, t' to S, S' are such that

$$t^2=S,\ t'^2=S,' \text{ and } tt'=\text{a constant, } b^2 \text{ say.}$$

Hence, the equation of the locus is

$$S.S'=b^4;$$

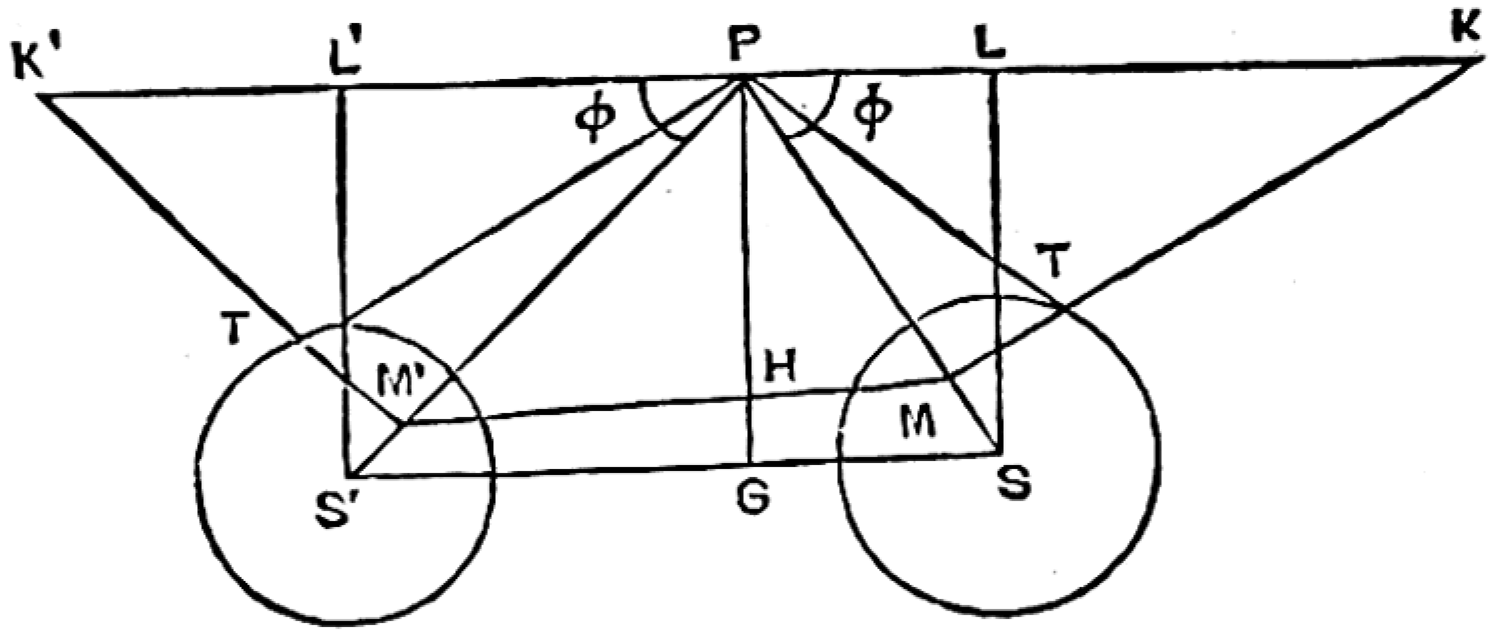
i.e., $$(r^2+2gx+k)\,(r^2+2g'x+k)=b^4. \quad \ldots \quad \ldots \ (1)$$

Equation (1) is included among the general class of bicircular quartics and is further capable of being written

$$[r^2+(g+g')x+k]^2-(g-g')^2x^2=b^4$$

or $$[r^2+(g+g')x+k]^2=(g-g')^2x^2+b^4 \quad \ldots \quad \ldots \ (2)$$

3. *If P be a point on the curve, straight lines drawn through T, T′ (the points of contact of tangents from P to S, S′) perpendicular to SP, S′P will meet the tangent at P in points equi-distant from P.*



Let the perpendiculars TM, T′M′ on SP, S′P′ meet the tangent at P in K, K′. Then, we have

$$t^2=PT^2=PM.PS=PL.PK$$
$$t'^2=PT'^2=PM'.PS'=PL.PK.'$$

Also, since $tt'=b^2$,

$$\frac{1}{t}.\frac{dt}{ds}+\frac{1}{t'}\frac{dt'}{ds'}=0.$$

and $$rdr=tdt\,;\ r'dr'=t'dt'. \quad \dots \quad \dots \quad \dots \quad (3)$$

Hence $$\frac{rdr}{t^2ds}+\frac{r'dr'}{t'^2ds'}=0\,;$$

i. e., $$\frac{1}{PM}\frac{dr}{ds}+\frac{1}{PM'}\frac{dr'}{ds'}=0.$$

But $$dr/ds=\cos\phi,-dr'/ds'=\cos\phi'.$$

∴ $$PM\sec\phi=P'M'\sec\phi',$$

or $$PK=P'K'.$$

Cor. $$r\cos\phi : r'\cos\phi'=t^2 : t'^2. \quad \dots \quad \dots \quad \dots \quad (4)$$

4. *If the normal at P meet SS′ in G, and MM′ in H, SG : S′G = $t^2 : t'^2$, and MH : M′H = PM^2 : PM′^{2}.*

For,

$$SG : S'G=PL : PL'$$
$$=PL.\ PK : PL.\ PK', \text{ since } PK=PK',$$
$$=t^2 : t'^2 \quad \dots \quad \dots \quad \dots \quad \dots \quad (5)$$

Also,

$$MH : M'H=PM\cos\phi : PM'\cos\phi'$$
$$=PM^2 : PM'^2, \text{ since } \cos\phi/\cos\phi'=PM/PM' \quad \dots \quad (6)$$

Cor. The normal is the symmedian through P of the triangle MPM′.

This follows from the well-known property of the symmedian of a triangle.

5. *Radius of Curvature at P.*

Let PSS$'=\theta$, PS$'$S$=\theta'$, PGS$=\psi$; then

$$\psi-\theta'=\tfrac{1}{2}\pi-\phi'\,;\quad \psi+\theta+\tfrac{1}{2}\pi-\phi=\pi$$

$$\therefore\qquad \frac{d\psi}{ds}-\frac{d\theta'}{ds}+\frac{d\phi'}{ds}=0\ ;\ \frac{d\psi}{ds}+\frac{d\theta}{ds}-\frac{d\phi}{ds}=0. \qquad \ldots\ (7)$$

Now, differentiating the relation (4)

$$r\cos\phi : r'\cos\phi'=t^2 : t'^2,$$

we have

$$\frac{1}{r}\frac{dr}{ds}-\tan\phi\frac{d\phi}{ds}-\frac{2}{t}\frac{dt}{ds}=\frac{1}{r'}\frac{dr'}{ds}-\tan\phi'\frac{d\phi'}{ds}-\frac{2}{t'}\frac{dt'}{ds};$$

Using (3), (7) and remembering that $rd\theta/ds=-\sin\phi$, $r'd\theta'/ds=\sin\phi'$, this may be written

$$\frac{1}{r}\cos\phi-\tan\phi\left(\frac{1}{\rho}-\frac{\sin\phi}{r}\right)-2\frac{r\cos\phi}{t^2}=-\frac{1}{r'}\cos\phi'-\tan\phi'\left(\frac{\sin\phi'}{r'}-\frac{1}{\rho}\right)+\frac{2\,r'\cos\phi'}{t'^2}.$$

Hence, simplifying,

$$\frac{1}{\rho}(\tan\phi+\tan\phi')=\frac{1}{r\cos\phi}+\frac{1}{r'\cos\phi'}-\frac{2r\cos\phi}{t^2}-\frac{2r'\cos\phi'}{t'^2};$$

i. e.,

$$\frac{1}{\rho}(\tan\phi+\tan\phi')=\frac{1}{\mathrm{PL}}+\frac{1}{\mathrm{PL'}}-\frac{4}{\mathrm{PK}}. \qquad \ldots\ \ldots\ (8)$$

6. The pedal equation with reference to the foot of the radical axis of S, S$'$ as origin, is found as follows :—

By § 2

$$\left.\begin{aligned} t^2&=r^2+2gr\cos\theta+k\\ t^2&=r^2+2g'r\cos\theta+k\\ tt'&=b^2.\end{aligned}\right\} \qquad \ldots\ \ldots\ \ldots\ (9)$$

If p denote the perpendicular from the origin O on the tangent at P,

$$p=r\sin\phi. \qquad \ldots\ \ldots\ \ldots\ (10)$$

Also, differentiating the first three equations (9) with respect to r and remembering that $rd\theta/dr=\tan\phi$, we have

$$t\frac{dt}{dr}=r+\frac{g\cos(\theta+\phi)}{\cos\phi};\ t'\frac{dt'}{dr}=r+\frac{g'\cos(\theta+\phi)}{\cos\phi};$$

and

$$\frac{1}{t}\frac{dt}{dr}+\frac{1}{t'}\frac{dt'}{dr}=0.$$

$$\text{Thus }\frac{1}{t^2}\left[r+\frac{g\cos(\theta+\phi)}{\cos\phi}\right]+\frac{1}{t'^2}\left[r+\frac{g'\cos(\theta+\phi)}{\cos\phi}\right]=0. \qquad \ldots\ (11)$$

Eliminating t, t', θ, ϕ from equations (9), (10), (11) we get the pedal equation.

As an example, consider the ordinary oval of Cassini. Here we may put

$$g=c=-g' ;\ k=c^2 ;$$

so that the above equations become

$$\begin{aligned} &t^2=r^2+2cr\cos\theta+c^2, \\ &t'^2=r^2-2cr\cos\theta+c^2, \\ &tt'=b^2, \text{ or } r^4+c^4-2c^2r^2\cos 2\theta=b^4, \\ &p=r\sin\phi, \\ &r(t^2+t'^2)+c\cos(\theta+\phi)\sec\phi\,(t'^2-t^2)=0. \end{aligned}$$

Substitute from the first two in the last and we get

$$2r(r^2+c^2)-4c^2(\cos\theta+\sin\theta\tan\phi)r\cos\theta=0.$$

i.e. $$\sqrt{(r^2-p^2)}\,\{(r^2+c^2)-2c^2\cos^2\theta\}=2c^2p\sin\theta\cos\theta.$$

$\therefore$ $$(r^2-p^2)(r^2-c^2\cos 2\theta)^2=c^4p^2(1-\cos^2 2\theta)$$

or $$p^2(c^4+r^4-2c^2r^2\cos 2\theta)=r^2(r^2-c^2\cos 2\theta)^2$$

or $$p^2.b^4=r^2(r^2-c^2\cos 2\theta)^2$$

or $$p\,b^2=r(r^2-c^2\cos 2\theta).$$

Hence $$\begin{aligned} 2rp.b^2&=2r^4-2c^2r^2\cos 2\theta \\ &=r^4+b^4-c^4 ; \end{aligned}$$

which is the pedal equation of the Cassinian. (*Vide*: Basset, § 251.)

M. T. NARANIENGAR.

Cyclotomic Equations.

1. If N be a prime number and a a number prime to N such that

$$a^e\equiv 1 \pmod{N},$$

e being the smallest power of a for which this is possible, then $(N-1)$ is divisible by e.

For, by Fermat's Theorem

$$a^{N-1}\equiv 1 \pmod{N}.$$

$\therefore$ (a^e-1) must be a factor of $a^{N-1}-1$.

i.e. $(N-1)$ must be divisible by e.

In other words, we may put $(N-1)$ equal to $e.f$.

2. Again, let b be a primitive congruence root of N, so that $(N-1)$ is the lowest power of b for which the congruence

$$b^{N-1}\equiv 1 \pmod{N}$$

is possible.

Take the terms of the series

$$\left.\begin{array}{ccccc} 1, & a, & a^2, & \ldots & a^{e-1} \\ b, & ba, & ba^2, & \ldots & ba^{e-1} \\ b^2, & b^2a, & b^2a^2, & \ldots & b^2a^{e-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b^{f-1}, & b^{f-1}a, & b^{f-1}a^2, & \ldots & b^{f-1}a^{e-1}. \end{array}\right\} \qquad (1)$$

(i) *No two terms of this set can be congruent to* N :
For if possible let

$$a^p b^q \equiv a^{p'} b^{q'}, \text{ where } q' > q.$$

Then $\qquad a^{p-p'} \equiv b^{q'-q}$, if $p > p'$;

otherwise $\qquad a^{e+p-p'} \equiv b^{q'-q}$, since $a^e \equiv 1$.

In other words, $a^{e'} \equiv b^{f'}$, where $e' < e$ and $f' < f$.

$\therefore \qquad a^{ee'} \equiv b^{ef'}$, or $b^{ef'} \equiv 1$,

which is contrary to the supposition, since $ef' < ef < N-1$.

Hence, no two terms of the above set are congruent with respect to N.

(ii) *Also,* $a^e \equiv bf'$; $b^f \equiv a^{e'}$.

For, the above set contains ef or $(N-1)$ terms, none of which are exactly divisible by N and the remainders are all different. Since there cannot be more than $(N-1)$ different remainders (zero being excluded) when the divisor is N, it follows that a^e and b^f should be congruent to one of the terms included in the above set. They cannot be congruent to $a^p.b^q$ since this would mean that

$$a^{e'} \equiv b^{f'} \, [e' < e, f' < f], \text{ which is impossible as proved in (i).}$$

Hence $\qquad a^e \equiv b^{f'}, \, [f' < f]$

and $\qquad b^f \equiv ae'. \quad [e' < e]$

3. *To solve the cyclotomic equation*

$$(x^N - 1)/(x-1) = 0.$$

Put

$$\left.\begin{array}{l} y_1 = \alpha + \alpha^a + \alpha^{a^2} \ldots\ldots\ldots\ldots\ldots\ldots + \alpha^{a^{e-1}} \\ y_2 = \alpha^b + \alpha^{ba} + \alpha^{ba^2} \ldots\ldots\ldots\ldots\ldots + \alpha^{ba^{e-1}} \\ y_f = \alpha^{b^{f-1}} + \alpha^{\quad f-1} + \ldots\ldots\ldots + \alpha^{b^{f-1}a^{e-1}} \end{array}\right\}$$

where α is a root of the cyclotomic equation.

It is easy to prove that the *y's* are the roots of a *rational* equation of the *f*th. degree. [*Cf*: Cajori: *Theory of Equations*, § 180.]

M. T. NARANIENGAR.

Differential Formulae in Astronomy.

The following differential formulae connecting the small variations of the parts of a spherical triangle are of fundamental importance in Astronomy:

$$\delta a = \cos B.\ \delta c + \cos C.\ \delta b + \sin h_a.\ \delta A \qquad \dots \qquad \dots (1)$$

$$\delta A = -\cos b\ \delta C - \cos c\ \delta B + \sin h_a.\ \delta a. \qquad \dots \qquad \dots (2)$$

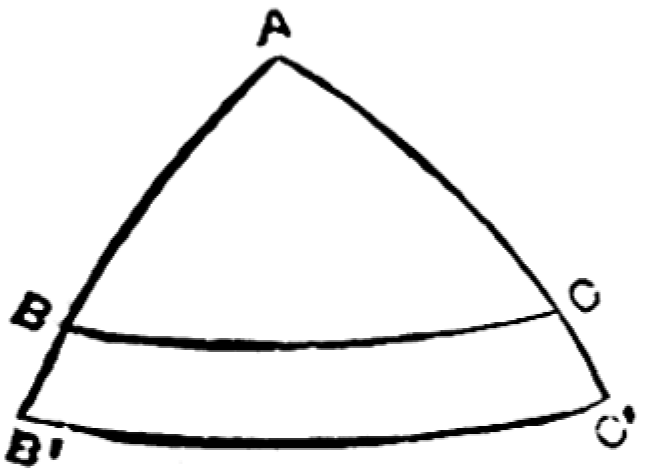
These are easily obtained analytically.

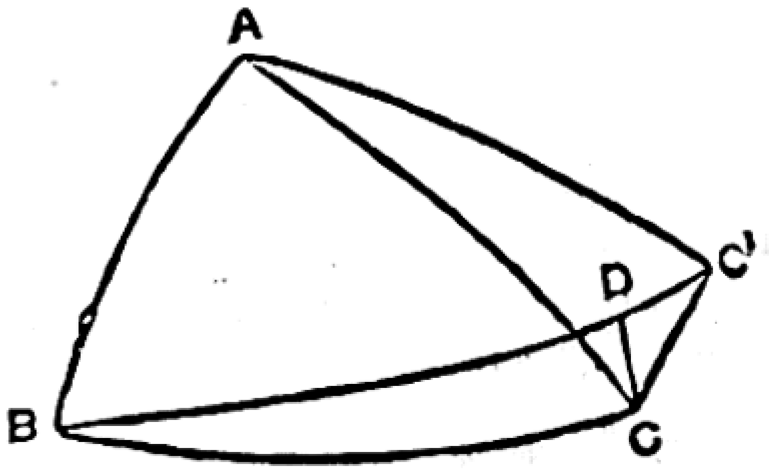
[*Vide*: (1) Simon Newcomb: *Sphl. Astro.*, § 6, and App. I, C.;
(2) W. Chavenet: *Astro.*, Vol. I., p. 50.]

The following geometrical proof may be of interest:—

First, let b and c alone vary in the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$





Project B′C′ on BC; then

(i) *Increment of* $BC = BB' \cos B + CC' \cos C$
$= \delta c \cos B + \delta b \cos C.$

Next, let A alone vary.

(ii) *Increment of* $BC = CC' \cos CC'B$
$= CC' \sin AC'B$
$= \sin b . \delta A . \sin C$
$= \sin h_a . \delta A,$

in the limit, since $AC'C = 90^\circ$ ultimately.

Hence when b, c, A all vary, *the total increment* of BC is

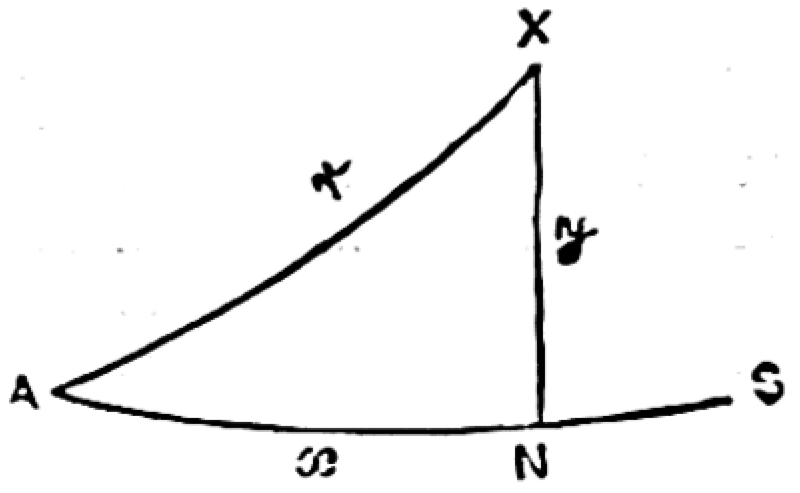
$$\delta a = \delta c \cos B + \delta b \cos C + \sin h_a\ \delta A,$$

which is formula (1).

A similar proof holds with regard to formula (2).

Example: To find the effects of aberration on the latitude and longitude of a star.

The effect of aberration is to give a star X a displacement XX′ towards a point A, 90° behind the sun S, such that



$$XX' = k \sin XA,$$

k being the constant of aberration.

Now in the triangle XAN, the angles A and N are unchanged by aberration and therefore by applying formula (1), we have

$$\delta y = \delta r \cos X + \delta x \cos N = \delta r \cos X; \ldots \qquad \ldots \quad (3)$$

$$\delta r = \delta y \cos X + \delta x \cos A, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

But $\delta r = -k \sin r, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$

$\therefore \quad \delta y = -k \sin r \cos X = -k \cos x \sin y, \ldots \quad \ldots \quad (6)$

since $\sin A = \cos X \sec x = \sin y \operatorname{cosec} r$.

Again, from (4), (5) and (6)

$$\begin{aligned} \delta x \cos A &= -k \sin r + k \sin r \cos^2 X \\ &= -k \sin r \sin^2 X \\ &= -k \sin x \sin X \\ &= -k \sin x \cos A \sec y, \end{aligned}$$

since $\sin X = \cos A \sec y = \sin x \operatorname{cosec} r$.

$\therefore \quad \delta x = -k \sin x \sec y. \quad \ldots \quad \ldots \quad \ldots \quad (7)$

Remembering that $90° - x = NS$ = difference of longitudes of the sun and the star, we get the usual expressions for the aberration in latitude and longitude from (6) and (7).

V. K. Aravamudan.

The Face of the Sky for May and June 1915.

Sidereal time at 8 p.m.

	May.				*June.*		
D.	H.	M.	S.		H.	M.	S.
1	10	33	42	... ...	12	35	55
8	11	1	18	... ...	13	3	31
15	11	28	54	... ...	13	31	7
22	11	56	30	... ...	13	58	43
29	12	24	6	... ...	14	26	19

From this table the constellations visible during the evening can be ascertained by a reference to their positions as given in the star-chart.

Phases of the Moon.

	May.				*June.*			
	D.	H.	M.		D.	H.	M.	
Last Quarter ...	6	10	53	A.M. ...	4	10	2	P.M.
New Moon ...	14	9	1	„ ...	13	0	27	A.M.
First Quarter ...	22	10	20	„ ...	20	7	54	P.M.
Full Moon ...	29	3	3	„ ...	27	9	57	A.M.

The Sun

enters Cancer (Solstice) on 22 June at 5-59 A.M.

The Planets.

Mercury is in superior conjunction with the Sun on May 1. Thereafter it is an evening star attaining its greatest elongation (23° 17′ E) on June 1. It is stationary on June 14 and is in inferior conjunction with the Sun on June 27. It is in conjunction with the Moon on May 15 and on June 14 and with Saturn on May 31.

Venus is a morning star. It is in conjunction with the Moon on May 11 and June 10 and with Mars on May 14.

Mars is in conjunction with the Moon on May 11 and June 9.

Jupiter is in quadrature on June 19. It is in conjunction with the Moon on May 9 and June 6.

Saturn is in conjunction with the Sun on June 28; it is in conjunction with the Moon on May 17 and June 14.

Uranus is in quadrature on May 7 and is stationary on May 21. It is in conjunction with the Moon on May 6, June 2, and June 30.

Neptune is in conjunction with the Moon on May 19 at 8-22 P.M., and June 16.

V. Ramesam.

SOLUTIONS.

Question 429.

(S. NARAYAN, B.A., L.T.) :—Through any point P in the plane of a triangle ABC, lines are drawn so that the intercepts made by the arms of each of the three angles may be bisected at P. Show that their six points of intersection with the three sides lie on a conic X which will be an ellipse, two parallel lines, or an hyperbola, according as P lies within, on, or without the maximum inscribed ellipse of ABC, and that X will be two intersecting lines through P, if P lies on the radical axis of the circumcircle and the polar circle of ABC.

Solution by N. Sankara Aiyar and 'Zero.'

Let P be the point (f, g, h); then the three bisected lines through P are obviously parallel to

$$\frac{\beta}{g}+\frac{\gamma}{h}=0,\ \frac{\gamma}{h}+\frac{\alpha}{f}=0,\ \frac{\alpha}{f}+\frac{\beta}{g}=0,$$

which are the harmonic conjugates of AP, BP, CP. Their equations are therefore of the forms

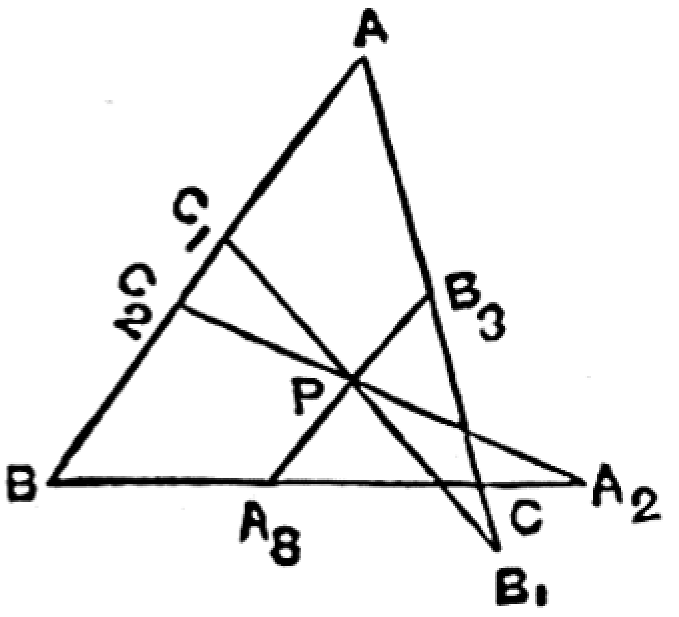
$$\lambda I+\beta/g+\gamma/h=0, \text{ \&c.}$$

where I denotes the lines at infinity, *viz* : $a\alpha+b\beta+c\gamma$. The value of λ is obtained from the condition that the line should pass through P. Thus

$$2\lambda k+2=0. \quad [2k\equiv af+bg+ch]$$

Hence the three bisected lines are

$$I-k\,(\beta/g+\gamma/h)=0, \text{ \&c.}$$



Also, evidently B_3C_2 is parallel to BC and its equation is

$$k\alpha-fI=0.$$

Any conic through the four points $B_1C_1B_3C_2$ is thus of the form

$$(k\alpha-fI)\,\{\,k(\beta h+\gamma g)-gh\,I\,\}+\lambda\beta\gamma=0$$

where λ is variable. If $\lambda=fk^2$, the conic becomes

$$k^2\Sigma(f\beta\gamma)-kI\,\Sigma(fg\alpha)+fgh\,I^2=0,$$

which by its symmetry must pass through A_2, A_3.

The line $I=0$ meets this where $\Sigma(f\beta\gamma)=0$. Hence the conic will be a hyperbola, parabola, or ellipse according as $\Sigma(af)^{\frac{1}{2}}$ is positive, zero, or negative; that is, according as (f, g, h) lies without, on, or, within the maximum inscribed ellipse $\Sigma(a\alpha)^{\frac{1}{2}}=0$.

If $\Sigma(af)^{\frac{1}{2}}=0$, $C_1\,B_3\,A_2$ and $C_2\,A_3\,B_1$ will be collinear. For, since B_1C_1 is $I-k(\beta/g+\gamma/h)=0$, the co-ordinates of C_1 are given by

$$I-k(\beta/g+\gamma/h)=0, \text{ and } \gamma=0.$$

i.e.,

$$a\alpha+(b-k/g)\,\beta=0, \text{ and } \gamma=0.$$

Similarly B_3, A_2 are given by

$$c\gamma+(a-k/f)\alpha=0,\ \beta=0\,;$$

and

$$b\beta+(c-k/h)\gamma=0,\ \alpha=0.$$

Hence, the condition of collinearity of $C_1\,B_3\,A_2$ is

$$abc=(a-k/f)(b-k/g)(c-k/h)\,;$$

which on simplification reduces to

$$k^3-k^2(af+bg+ch)+k(abfg+bcgh+cahf)=0\,;$$

or

$$(af+bg+ch)^2=4(ab\,fg+\ldots\ldots),$$

or

$$\Sigma(af)^{\frac{1}{2}}=0,$$

since $2k=af+bg+ch$.

Question 438.

(A. C. L. Wilkinson, M.A., F.R.A.S.):—If through a point P in the plane of a triangle ABC, straight lines are drawn parallel to the sides, show that the six points of intersection with the sides lie on a conic which is an ellipse, parabola, or hyperbola according as P lies within, on, or outside the maximum inscribed ellipse of ABC. Shew also that it is two straight lines if P lies on the minimum ellipse circumscribing ABC. (Suggested by Q. 429.)

Solution by N. Sankara Aiyar and Zero.

The equations of the lines through P parallel to sides of ABC are evidently

$$fI-2k\alpha=0, \text{ etc.}$$

The six points of intersection of these with the sides therefore lie on

$$(fI-2k\alpha)(gI-2k\beta)(hI-2k\gamma)=-8k^3\,\alpha\beta\gamma,$$

which reduces to the conic

$$fgh\, I^2 - 2k\, I\, \Sigma(\alpha gh) + 4k^2 \Sigma(f\beta\gamma) = 0.$$

As in Q. 429, this is a hyperbola, parabola or ellipse according as P is without, on, or within the maximum inscribed ellipse.

The condition for two straight lines is obtained directly as in Question 429, and is found to be

$$(af-2k)\,(bg-2k)\,(ch-2k) = abcfgh\,;$$

or

$$4k^2 - 2k\,\Sigma(af) + \Sigma(abfg) = 0\,;$$

i.e.

$$\Sigma(ab\,fg) = 0,$$

which is identical with the condition that the point P (f, g, h) should lie on the minimum circumscribed ellipse of the triangle ABC.

Question 480.

(**V. Ramaswami Aiyar**) :—If a straight line cut a three-cusped hypocycloid at P, Q, R, S, show that the extremities of the tangent chords at P, Q, R, S, lie on a rectangular hyperbola.

If $S=0$, $S'=0$ be two quartics then $S+k\,S'=0$ is the equation of the family of quartics passing through the sixteen points of intersection of $S=0$ and $S'=0$.

Let $S=0$ denote the tricusp $\alpha^{-\frac{1}{2}}+\beta^{-\frac{1}{2}}+\gamma^{-\frac{1}{2}}=0$; and $S'=0$ denote the four tangent chords, *viz.*, $\alpha x^3_1+\beta y^3_1+\gamma z_1^3=0$, etc., such that $(x_1\, y_1\, z_1)$ etc., lie on the given line $l\alpha+m\beta+n\gamma=0$.

Hence we have to prove that $S+k\,S'=0$ may for a suitable value of k be made to represent the line $l\alpha+m\beta+n\gamma=0$ twice, and a conic on which lie the eight points of intersection of the tangent chords with the tricusp.

In other words

$$\Sigma\alpha^2\beta^2 - 2\alpha\beta\gamma\Sigma(\alpha) + k\Pi(\alpha x_1^3+\beta y_1^3+\gamma z_1^3)$$
$$\equiv (l\alpha+m\beta+n\gamma)^2\,\{\,\Sigma(u\alpha)^2 + 2\Sigma(u'\beta\gamma)\,\}.$$

Equating coefficients of like terms, we get

$$ul^2 = kx_1^3x_2^3x_3^3x_4^3,\quad vm^2 = ky_1y_2^3y_3^3y_4^3;\ \text{etc.} \qquad \dots\ (1)$$
$$2ulm + 2u'l, = k\Sigma(x_1^3y_2^3x_3^3x_4^3)\ ;\ \text{etc.} \qquad \dots\ (2)$$

But $x_1x_1x_3x_4 : y_1y_2y_3y_4 : z_1z_2z_3z_4 = l : m : n.$ (J.I.M.S., Vol. V, p. 84)

$$\therefore\ \frac{ul^2}{l^3} = \frac{vm^2}{m^3} = \frac{wn^2}{n^3}\,;\ i.e.,\ u : v :\ w = l : m : n.$$

Again
$$\Sigma\,\{\,x_1^3y_2^3x_3^3x_4^3\,\} = x_1^3x_2^3x_3^3x_4^3\Sigma\left(\frac{y_2^3}{x_2^3}\right),$$

and
$$\Sigma\frac{y_2^3}{x_2^3} = \left(\ \Sigma\frac{y_2}{x_2}\right)^3 - 3\Sigma\frac{y_2}{x_2}.\Sigma\frac{y_2y_3}{x_2x_3} + 3\Sigma\frac{y_2y_3y_4}{x_2x_3x}.$$

The equation giving $\frac{y_2}{x_2}$ is, (J. I. M. S., Vol. V, p. 84),

$$ly^4+2lxy^3+x^2y^2(l+m+n)+2mx^3y+mx^4=0.$$

$$\therefore \qquad \Sigma\frac{y_2^3}{x_2^3}=-8+3\cdot2\cdot\frac{l+m+n}{l}-\frac{6m}{l}=\frac{6n}{l}-2.$$

$$\therefore \qquad 2ulm+2u'\,l^2=kx_1^3x_2^3x_3^3x_4^3\left(\frac{6n}{l}-2\right)$$
$$=ul^2\left(\frac{6n}{l}-2\right).$$

$$\therefore \qquad 2u'l^2=ul\,(6n-2l-2m).$$
$$\therefore \qquad u':u=3n-l-m:l.$$

Hence the required conic may be written in the form

$$\Sigma(l\alpha^2)+2\Sigma(3l-m-n)\,\beta\gamma=0.$$

This is a rectangular hyperbola, in as much as

$$\Sigma(l)-2\Sigma(3l-m-n)\cos A=\Sigma(l)-\Sigma(3l-m-n)=0,$$

since $2\cos A=2\cos B=2\cos C=1$.

Hence the eight points in question lie on a rectangular hyperbola.

Question 483.

(V. V. S. NARAYAN):—Construct a triangle ABC, being given the angle A, the side AB and the distance of the middle point of AC from the symmedian point.

Geometrical Solution by the Proposer.

Given a side and an angle adjacent to it, the locus of the Lemoine point is a conic (Neuberg). Hence, since the angle A and AB are given the conic or the locus of K can be drawn. Take AB and draw BAC so that BAC$=\angle$A. From B draw BD$\perp^r$ to AC and let M be its middle point. Through M draw MPK to cut the conic at P and AC at K so that PK is of given length. Set off KC along AC so that KC$=$KA. Join ABC which shall be the required triangle.

Question 523.

(R. N. APTE, M.A., F.R.A.S.) :—Find the value of

$$\iint xyz\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}dxdy$$

where $z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$ and the integration extends over the positive quadrant of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Additional Solution by T. P. Bhaskara Shastri.

Here $\frac{dz}{dx}=-\frac{c^2x}{a^2z}$; $\frac{dz}{dy}=-\frac{c^2y}{b^2z}$.

The given integral U

$$=c\iint xy\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{-\frac{1}{2}}\left\{1+\frac{c^2-a^2}{a^4}x^2+\frac{c^2-b^2}{b^4}.y^2\right\}dxdy,$$

where the integration extends to all positive values of x and y subject to the condition $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2<1$.

$$\therefore\quad U=c\iint xy\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}}dxdy+\frac{c^3}{a^4}\iint x^3y\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{-\frac{1}{2}}dx\,dy$$
$$+\frac{c^3}{b^4}\iint xy^3\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{-\frac{1}{2}}dxdy.$$

By Dirichlet's theorem,

$$U=c.\frac{a^2b^2}{4}\cdot\frac{\Gamma(1)\Gamma(1)}{\Gamma(2)}\int_0^1(1-u)^{\frac{1}{2}}udu$$
$$+\frac{c^3}{a^4}\cdot\frac{a^4b^2}{4}\frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}\int_0^1(1-u)^{-\frac{1}{2}}u^2du$$
$$+\frac{c^3}{b^4}\cdot\frac{a^2b^4}{4}\frac{\Gamma(1)\Gamma(2)}{\Gamma(3)}\cdot\int_0^1(1-u)^{-\frac{1}{2}}u^2du.$$

Now $$\int_0^1 u(1-u)^{\frac{1}{2}}du=\frac{4}{15}.$$

$$\int_0^1 u^2(1-u)^{-\frac{1}{2}}du=\frac{16}{15}.$$

$$\therefore\quad U=\frac{ca^2b^2}{4}\cdot\frac{4}{15}+\frac{16}{15}\cdot\frac{c^3b^2}{4}\cdot\frac{1}{2}+\frac{16}{15}\cdot\frac{c^3a^2}{4}\cdot\frac{1}{2}$$
$$=\frac{c}{15}\left[a^2b^2+2c^2(a^2+b^2)\right].$$

Question 532.

(R. VYTHYNATHASWAMY) :—If 2A, 2B, 2C are the mutual inclinations of any three lines in space and 2L, 2M, 2N, the angles which any fourth straight line makes with them, shew that the following identical relation holds

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & \sin^2 C & \sin^2 B & \sin^2 L & 1 \\ 1 & \sin^2 C & 0 & \sin^2 A & \sin^2 M & 1 \\ 1 & \sin^2 B & \sin^2 A & 0 & \sin^2 N & 1 \\ 1 & \sin^2 L & \sin^2 M & \sin^2 N & 0 & 1 \\ 4 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

Remarks by the Proposer.

Consider the product of the rectangular arrays

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & l_1\sqrt{2} & m_1\sqrt{2} & n_1\sqrt{2} & 1 \\ 1 & l_2\sqrt{2} & m_2\sqrt{2} & n_2\sqrt{2} & 1 \\ 1 & l_3\sqrt{2} & m_3\sqrt{2} & n_3\sqrt{2} & 1 \\ 1 & l_4\sqrt{2} & m_4\sqrt{2} & n_4\sqrt{2} & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & l_1\sqrt{2} & m_1\sqrt{2} & n_1\sqrt{2} & 1 \\ 1 & l_2\sqrt{2} & m_2\sqrt{2} & n_2\sqrt{2} & 1 \\ 1 & l_3\sqrt{2} & m_3\sqrt{2} & n_3\sqrt{2} & 1 \\ 1 & l_4\sqrt{2} & m_4\sqrt{2} & n_4\sqrt{2} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

where the l's, m's, n's are direction cosines. The product vanishes identically. The product is easily seen to reduce to the given determinant.

Or we may proceed as follows: From a point O draw line parallel to the given lines and measure along them OA, OB, OC, OD each equal to unity; and write down the identical relation between the distances of the five points O, A, B, C, D expressing the distances in terms of the mutual angles. The result will be that stated in the question.

A simpler relation between the mutual inclinations of any four lines in space is that given in Routh's *Analytical Statics, viz.*

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \cos(14) \\ \cos(21) & 1 & \cos(23) & \cos(24) \\ \cos(31) & \cos(32) & 1 & \cos(34) \\ \cos(41) & \cos(42) & \cos(43) & 1 \end{vmatrix} = 0,$$

which is easily obtained by considering the product of the rectangular array

$$\left|\begin{array}{ccc} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_4 & m_4 & n_4 \end{array}\right|$$

by itself.

Question 534.

(Zero) :—A given ellipse is projected orthogonally on a given plane. Shew how to obtain the axes of the projection geometrically.

Solution by A. Narasinga Rao and the Proposer.

Construct an ellipse concentric with the given ellipse, having its minor axis parallel to the common section of the two planes, and the ratio of its axes equal to the cosine of the inclination of the planes. Such an ellipse projects into a circle and therefore the common diameters of the two ellipses project into equal diameters of the projection in question. The bisectors of the angles between these equal diameters will give the axes of the projection.

If we take the limiting cases of the concentric ellipse having double contact with the given ellipse internally and externally, we obtain the axes of the projection in position and magnitude. For, the projections of these limiting ellipses will be concentric circles having double contact with the projection in question and the diameters of contact are therefore the axes in magnitude and position.

Question 537.

(K. J. Sanjana, M. A.) :—Prove that

$$\frac{1}{2}\,\frac{x^2}{2}-\frac{1\cdot3}{2\cdot4}\,\frac{x^4}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{6}-\dots\dots\dots$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{8}\log(1+x_1^2)+\frac{1}{16}\log(1+x_2^2)+\dots\dots$$

where $$x_1^2=\frac{x^4}{4(x^2+1)},\quad x_2^2=\frac{x_1^4}{4(x_1^2+1)};\quad\dots\dots$$

and that

$$\frac{1}{2}\,\frac{x^2}{2}+\frac{1\cdot3}{2\cdot4}\,\frac{x^4}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{6}+\ldots\ldots$$

$$=-\frac{1}{4}\log(1-x^2)-\frac{1}{8}\log(1+x_1^2)-\frac{1}{16}\log(1+x_2^2)-\ldots\ldots$$

where $$x_1^2=\frac{x^4}{4(1-x^2)},\quad x_2^2=\frac{x_1^4}{4(x_1^2+1)},\ \ldots\ldots$$

In the first equality x is any real quantity, in the second any real quantity less than 1.

Additional Solution by N. Sankara Aiyar, M. A.

Now

$$(1+x^2)^{-\frac{1}{2}}=1-\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}.\,x^4-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6\ldots\ldots$$

$$\therefore\qquad\int\frac{(1+x^2)^{-\frac{1}{2}}}{x}dx=\log x-\frac{1}{2}\,\frac{x^2}{2}+\frac{1\cdot3}{2\cdot4}\,\frac{x^4}{4}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{6}+\ldots\ldots$$

Thus the first series is equal to $\log x+c-\int\frac{(1+x^2)^{-\frac{1}{2}}}{x}dx$

$$=\log x+c+\log\frac{1+\sqrt{1+x^2}}{x}=c+\log(1+\sqrt{1+x^2}).$$

But, since the given series vanishes with x, we have

$$c+\log 2=0.$$

Hence the given series is equal to

$$\log(1+\sqrt{1+x^2})-\log 2$$

$$=\frac{1}{2}\log(1+1+x^2+2\sqrt{1+x^2})-\log 2$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{2}\log\frac{x^2+2+2\sqrt{1+x^2}}{2\sqrt{1+x^2}}-\frac{1}{2}\log 2$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{2}\log\left(1+\frac{x^2+2}{2\sqrt{1+x^2}}\right)-\frac{1}{2}\log 2.$$

Let $$\frac{x^2+2}{2\sqrt{1+x^2}}=\sqrt{1+x_1^2}.$$

$$\therefore\qquad\frac{x^4+4x^2+4}{4(1+x^2)}=4(1+x_1^2).$$

$$\therefore\qquad x_1^2=\frac{x^4}{4(1+x^2)}$$

Thus the given series is equal to

$$\tfrac{1}{4}\log(1+x^2)+\tfrac{1}{2}\log(1+\sqrt{1+x_1^2})-\tfrac{1}{2}\log 2$$

$$=\tfrac{1}{4}\log(1+x^2)+\tfrac{1}{8}(1+x_1^2)+\tfrac{1}{4}\log(1+\sqrt{1+x_2^2})-\tfrac{1}{4}\log 2;$$

and the transformation can be continued *ad infinitum*.

Next $(1-x^2)^{-\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}x^4+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot x^6\ldots\ldots\ldots\ldots$

and the second series $=\int\frac{(1-x^2)^{-\frac{1}{2}}}{x}dx+c-\log x.$

$$=\log\frac{1-\sqrt{1-x^2}}{x}+c-\log x.$$

$$=\log\frac{1-\sqrt{1-x^2}}{x^2}+c$$

$$=\log\frac{1}{1+\sqrt{1-x^2}}+c.$$

$$=-\log(1+\sqrt{1-x^2})+\log 2\ [\text{putting } x=0,$$

$$=-\tfrac{1}{2}\log(2-x^2+2\sqrt{1-x^2})+\log 2.$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{2}\log\frac{2-x^2+2\sqrt{1-x^2}}{2\sqrt{1-x^2}}+\tfrac{1}{2}\log 2.$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{2}\log\left(1+\frac{2-x^2}{2\sqrt{1-x^2}}\right)+\tfrac{1}{2}\log 2$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{2}\log(1+\sqrt{1+x_1^2})-\tfrac{1}{2}\log 2,\text{ as before;}$$

and the transformation can be continued *ad infinitum.*

Question 557.

(PROFESSOR SANJANA) :—At a point P of a parabola the normal chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that PQ. AN=2 SP. PG.

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$SP_1.SP_2=AS.SP$, $PP_1.PP_2=PQ.AN$, where S is the focus.

Shew also that ths circle P_1P_2S passes through the circumcentre of PP_1P_2.

Note by V. V. S. Narayan.

"If from any point three normals are drawn to a parabola the circle through their feet passes through the vertex of the parabola." Add to this the following property ennunciated by Mr. M. T. Naraniengar in the Mathematical columns of the *Educational Times* :—"If the normal at P meets the curve at Q and the normal at Q meets the axis in G, then shall APGQ be cyclic." We see that in the present case $GPAP_1P_2$ are concyclic. Now, to prove $SP_1\cdot SP_2=AS\cdot SP$, produce P_1S to cut the circle $GPAP_1P_2$ at R. Then $RS=SP_2$, since $P_1SP_2=2P_1RP_2$,

$$\therefore\quad SP_1\cdot SP_2=SP_1\cdot SR=AS\cdot SB=AS\cdot SP.$$

Similarly all the other parts of the question can be proved.

The properties ennnciated in the question follow also from the following theorem fully worked out on page 263 of Milne's *Weekly Problem Papers*:

If OP, OQ, OR, the three normals passing through O, cut the parabola again in P′, Q′, R′, then

(i) $$OP'.OQ'.OR' = OD^2.OD'^2.OS/ON^2$$

where D, D′, N are points where the $\perp^r$ from O on the axis cuts the curve and the axis;

(ii) $$\frac{OP.OQ.OR}{OP'.OR'.OR'} = \frac{ON^2}{OD.OD'}.$$

In the present case O is on the curve and consequently, the three normals are PQ_1, PP_1,PP_2 and applying the above two properties the results stated in the question immediately follow. Milne's solution is, however, *analytical.*

Question 563.

(N. P. Pandya) :—Given two quadrilaterals in the plane of the paper, shew how to draw a straight line bisecting them both.

Solution by A Narasinga Rao.

Let ABCD be one of the quadrilaterals. Produce AB and CD to meet at E.

If PQ be a line bisecting ABCD, P and Q lying in AB and CD respectively the envelope of PQ is a hyperbola H_1 having EB, EC for asymptotes.

Also, the points of contact of all tangents like PQ which cut AB. CD internally lie on a certain arc of this hyperbola.

Similarly all lines PQ which cut AD, BC internally and bisect ABCD touch an arc of a hyperbola H_2 having AD, BC for asymptotes.

In the same way, from the second quadrilateral we obtain two hyperbolas h_1, h_2 the tangents to which bisect the quadrilateral.

A common tangent of H and h with its points of contact in prescribed limits is a solution of the problem.

QUESTIONS FOR SOLUTION.

Correction: In Q. 624 *insert at the end*—'where t, s denote tan α, tan β respectively.'

630. (S. NARAYANA AIYAR, M.A.) :—If

$$A_n=\frac{(a-b)(a-bx)(a-bx^2)(a-bx^3)\ldots\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \ldots\ldots\ldots\ (1-x^n)}$$

and
$$B_n=\frac{(b-a)(b-ax)(b-ax^2)(b-ax^3)\ldots\ldots\ldots(b-ax^{n-1})}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \ldots\ldots\ldots\ (1-x^n)},$$

shew that

$$(-1)^n B_n=\begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ A_2 & A_1 & 1 & 0 & 0 & \ldots & 0 \\ A_3 & A_2 & A_1 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n-1} & A_{n-2} & A_{n-3} & \ldots & \ldots & \ldots & 1 \\ A_n & A_{n-1} & A_{n-2} & \ldots & \ldots & \ldots & A_1 \end{vmatrix}$$

with a similar determinant for $(-1)^n A_n$ in terms of the constituents $B_1, B_2, B_3\ldots\ldots\ldots B_n$.

631. (S. NARAYANA AIYAR, M.A.) :—Shew that if $\phi\,(a,b,c,d)$ denotes

$$1+\frac{x}{1}\cdot\frac{c-d}{1}\cdot\frac{b+x-1}{a+x-1}+\frac{x(x-1)}{1\cdot 2}\cdot\frac{(c-d)(c-d-1)}{d(d+1)}\cdot\frac{(b+x-1)(b+x-2)}{(a+x-1)(a+x-2)}+\ldots$$

then
$$\frac{\phi(a,\,b,\,c,\,d)}{\phi(c,\,d,\,a,\,b)}=\frac{\Gamma(x+b)\Gamma(x+c)\Gamma(a)\Gamma(d)}{\Gamma(x+a)\Gamma(x+d)\Gamma(b)\Gamma(c)}.$$

632. (R. N. APTE, M.A., F.R.A.S.) :—If r_1, r_2, $r_3\ldots r_n$ be the distances of a point P in the plane of a regular polygon from the vertices, find $\Sigma(r^2_1 r^2_2 r^2_3)$ in terms of n, a the radius of the circumcircle of the polygon and c the distance of P from the centre of the circle, the summation extending over all possible groups of three vertices.

633. (K. APPUKUTTAN ERADY, M.A.) :—A particle is projected from a point on the boundary of a circular pond of radius a with the velocity

$\sqrt{2gc}$. Prove that, if all directions of projection are equally probable, the chance of its falling into the pond is

$$\frac{1}{2}-\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{a}{c}\cos\phi} \quad d\phi$$

634. (E. H. Neville, B.A., B.Sc.) :—On the section of a surface by a plane through the normal ON at a point O is taken a point S at a small distance f from O, and the plane parallel to the tangent plane at O and distant $1/\lambda$ from O meets the normal at O and S to the surface in points L and P. Shew that LP² is approximately $(f^2/\lambda^2)\{(k_n-\lambda)^2+a_g\}$, where k_n, a_g are the normal curvature and geodesic torsion of the surface along OS; deduce that if S traces a small contour round O, λ can be chosen so that P traces a curve which is almost a circle with centre L only if the contour traced by S approximates either to a circle with centre O or to an ellipse whose axes are the principal tangents to the surface, and that λ in the first case must be the mean curvature of the surface but in the second case may have either of two values.

635. (S. Krishnasawmi Aiyangar, M.A.) :—If $\sigma_1, \sigma_2, \sigma_3$ represent KA, KB, KC where K is the symmedian point, and m_1, m_2, m_3 the medians, of a triangle ABC, show that

1. $\Sigma\{a(b^2+c^2)\,\sigma_1\,m_1\} = \frac{1}{2}\,abc\,(a^2+b^2+c^2)+\frac{3abc}{2}\cdot\frac{a^4+b^4+c^4}{a^2+b^2+c^2}$.

2. $\Sigma a(b^2+c^2)\frac{\sigma_1}{m_1}=4abc.$

3. $\Sigma bc\frac{m_1}{\sigma_1}=\frac{3}{2}(a^2+b^2+c^2)$.

636. (S. Krishnaswami Aiyangar, M.A.) :—If

$$\sin e^x=\sum_0^\infty a_n\frac{x^n}{\underline{|n}},\quad \text{and } \cos e^x=\sum_0^\infty b_n\frac{x^n}{\underline{|n}},$$

show that
$$a_n+a_{n+1}\log p+a_{n+2}\frac{(\log p)^2}{2!}+\dots\dots\dots$$
$$=p-3^n\frac{p^3}{3!}+5^n\frac{p^5}{5!}-\dots\dots\dots$$

and find $\sin(e^x+e^{-x})$.

637. (D. Krishnamurti) :—Solve

$$x\left(\frac{d^2y}{dx^2}-y\right)=1+\frac{dy}{dx}.$$

638. (S. B. Belekar):—If $\lambda=e^{i}\alpha$, show that the point

$$\frac{a(\lambda^2+1)}{2\lambda}, \frac{b(\lambda^2-1)}{2i\lambda}$$

lies on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If λ_1, λ_2, λ_3 be three points as above, the circle passing through these three is

$$x^2+y^2-\frac{(a^2-b^2)}{4a\mathrm{N}}\{\mathrm{N}^2+\mathrm{M})+(\mathrm{LN}+1)\}x-$$

$$\frac{(a^2-b^2)}{4i\mathrm{LN}}\{(\mathrm{N}^2+\mathrm{M})-(\mathrm{LN}+1)\}y$$

$$+\frac{(\mathrm{MN}+\mathrm{L})(a^2-b^2)-2(a^2+b^2)\mathrm{N}}{4\mathrm{N}}=0,$$

where $\mathrm{L}\equiv\Sigma\lambda_1$, $\mathrm{M}\equiv\Sigma\lambda_1\lambda_2$, $\mathrm{N}\equiv\lambda_1\lambda_2\lambda_3$.

639. (V. V. S. Narayan) :—The polar circles of the five triangles external to a pentagon, which are formed by producing its sides have a common orthogonal circle.

640. (Selected) :—

The Magic Number Cards.

"This trick consists of six cards with certain of the numbers from 1 to 63 printed on them. The magician asks you to think of a number between 1 and 63 and then presents these cards one at a time with the request that you indicate whether your number is on the card. This done he at once declares the number which you thought of."

Explain the trick and point out any easy extensions of the trick.

641. (Selected) :—"There cannot be five prime numbers in arithmetical progression unless their common difference be divisible by 2×3×5=30, except when the first term of the progression is 5."

Barlow's *Theory of Numbers*, p. 65.

Illustrate or criticise the above statement.

642. (S. Ramanujan):—Shew that

$$(1)\ \sum_{n=0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{2n+1}\right)5^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{4\sqrt{5}}-\frac{\sqrt{5}}{24}[\log(2+\sqrt{5})]^2$$

$$(2)\ \sum_{n=0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{2n+1}\right)9^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{8}-\frac{3}{8}[\log 2]^2.$$

643. (K. V. Ananthanarayana Sastri, B.A.):—Prove that

$$\frac{1}{2n+2}+\frac{1}{2}\,\frac{1}{(2n+4)}+\frac{1\cdot3}{2\cdot4}\,\frac{1}{(2n+6)}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{1}{(2n+8)}+\ldots=\frac{2\cdot4\cdot6\ldots2n}{3\cdot5\cdot7\cdot\ldots(2n+1)}.$$

644. (K. V. Ananthanarayana Sastri, B.A.):—Shew that the length of the fourth positive pedal of a loop of the Lemniscate of Bernoulli is given by

$$18a\int_0^1\frac{x^8\,dx}{\sqrt{(1-x^4)}}.$$

645. (R. P. Vythynathaswamy):—Shew that an identical relation must exist between the mutual moments of any seven lines in space and find it in its determinental form.

About how many given lines should the moment of a line be known to determine the latter completely.

List of Periodicals Received.

(From 16th January to 15th March 1915.)

1. Annals of Mathematics, Vol. 16, No. 2, December 1914.
2. Astrophysical Journal, Vol. 40, Nos. 4 & 5, November and December 1914.
3. Bulletin of the American Mathematical Society, Vol. 21, No. 4, January 1915.
4. Bulletin des Sciences Mathematiques, Vol. 38, November and December 1914.
5. Educational Times, January and February 1915, (6 Copies.)
6. L'Intermediaire des Mathematiciens, Vol. 21, Nos. 9 & 10, September and October 1914.
7. Mathematical Gazette, Vol. 7, No. 114, December 1914, (4 copies).
8. Mathematical Reprints from Educational Times, Vol. 26, (2 Copies.)
9. Mathematics Teacher, Vol. 7, No. 2, December 1914.
10. Messenger of Mathematics, Vol. 44, Nos. 7 & 8, November and December 1914.
11. Monthly Notices of the Royal Astronomical Society Vol. 75 Nos. 1 & 2, November and December 1914.
12. Philosophical Magazine, Vol. 29, Nos. 169 & 170, January and February 1915.
13. Popular Astronomy, Vols. 22 & 23 Nos. 10 & 1, December 1914 and January 1915, (3 copies).
14. Proceedings of the London Mathematical Society, Vol. 14, No. 2, January 1915.
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A Special Class of Equations.

By M. T. Naraniengar.

Consider the equations

$$\phi(x)-x=0,\ \phi\{\phi(x)\}-x=0,\ \phi[\phi\{\phi(x)\}]-x=0, \text{ etc.}$$

These may conveniently be written

$$\phi(x)-x=0,\ \phi^2(x)-x=0,\ \phi^3(x)-x=0, \text{ etc.}$$

We shall call the *first* equation the *primary* equation of the class, and the rest *secondary* equations.

Theorem I:—Every root of the primary equation is a root of a secondary equation.

For, if α is a root of the primary,

$$\phi(\alpha)=\alpha.$$

$$\therefore \qquad \phi^2(\alpha)=\phi[\phi(\alpha)]=\phi(\alpha)=\alpha\ ;$$

or α is also a root of $\phi^2(x)-x=0$; and so on.

Cor. Any root of $\phi^n(x)-x=0$, is also a root of $\phi^{mn}(x)-x=0$.

Theorem II:—If m and n are prime to each other, the equations $\phi^m(x)-x=0$, and $\phi^n(x)-x=0$ have no roots common other than those of the primary.

Find integers a, b such that $am-bn=\pm 1$, which is always possible since m and n are prime to each other.

Now, if α be a common root of the two secondaries,

$$\phi^{am}(\alpha)=\alpha \text{ and } \phi^{bn}(\alpha)=\alpha,$$

supposing $am-bn=1$,

$$\phi^{am}(\alpha)=\phi^{bn+1}(\alpha)=\phi\{\phi^{bn}(\alpha)\}=\phi(\alpha).$$

$$\therefore \qquad \phi(\alpha)=\alpha.$$

Hence α is a root of the primary.

Theorem III:—If h is the greatest common factor of m and n, the roots of $\phi^h(x)-x=0$, are all the common roots of $\phi^m(x)-x=0$ and $\phi^n(x)-x=0$.

For, in this case, we can find a, b such that

$$am-bn=\pm h,$$

so that if α is a common root,

$$\phi^{am}(\alpha)=\alpha \text{ and } \phi^{bn}(\alpha)=\alpha.$$

Thus $\phi^{am}(\alpha)=\phi^{bn+h}(\alpha)=\phi^h\{\phi^{bn}(\alpha)\}=\phi^h(\alpha)$.

$$\therefore \qquad \phi^h(\alpha)=\alpha.$$

Hence the theorem.

Theorem IV:—If n be a prime number and α be a primitive root of $\phi^n(x)-x=0$, [that is, a root other than the roots of the primary equation], then $\phi(\alpha)$, $\phi^2(\alpha)$,......$\phi^{n-1}(\alpha)$ are also roots and are all different.

Let $\beta_p\equiv\phi^p(\alpha)$, $[p=1, 2,\ldots n-1]$. Then we have

$$\phi^n(\beta_p)=\phi^n\{\phi^p(\alpha)\}=\phi^p\{\phi^n(\alpha)\}=\phi^p(\alpha)=\beta_p.$$

Hence β_p is a root of the equation $\phi^n(x)-x=0$.

Also β_p cannot be equal to β_q, since this would mean

$$\phi^p(\alpha)=\phi^q(\alpha),$$

i.e.
$$\phi^{p-q}(\beta_q)=\beta_q,\ (p>q).$$

In other words, β_q is a common root of $\phi^{p-q}(x)-x=0$ and $\phi^n(x)-x=0$, which is contrary to Theorem II since $(p-q)$ is prime to n.

Theorem V :—If n is a prime number, then

$$[\phi^n(x)-x]\equiv\{\phi(x)-x\}.\prod\{f(x)\}$$

where $f(x)$ is a cyclic expression of the n^{th} degree.

Let $\phi(x)-x=0$ be of degree p. Then the equation $\phi^n(x)-x=0$ is of degree p^n and has (p^n-p) primitive roots, being the roots of

$$\{\phi^n(x)-x\}\div\{\phi(x)-x\}=0. \quad \dots \quad \dots \quad (1)$$

But, by Theorem IV, a primitive root α has $(n-1)$ other roots associated with it, *viz.*: $\phi(\alpha)$, $\phi^2(\alpha)$,...$\phi^{n-1}(\alpha)$. Thus the set of roots corresponding to a primitive root (α) belong to a cyclic equation $f(x)=0$ of degree n.

Hence, the left hand side of (1) is made up of factors of the type $f(x)$, which is the result stated in the theorem.

Cor. 1. *Fermats' Theorem follows at once from the above.*

For
$$p^n-p\equiv0,\ (\text{mod } n).$$

If p and n be prime to each other, therefore,

$$p^{n-1}-1\equiv0,\ (\text{mod } n),$$

which is Fermat's Theorem.

Cor. 2. $f\{\phi(x)\}\equiv(-1)^{p-1}f(x_1).f(x_2)\dots\dots f(x_p)$, where $x_1,x_2,\dots x_p$ are the p values of X in the equation

$$\phi(X)-\phi(x)=0.$$

For, from the cyclic property of $f(x)$ we readily see that $f\{\phi(x)\}$ is divisible by $f(x)$; so that, the left hand member is divisible by the product $f(x_1), f(x_2)\dots\dots f(x_p)$. Also, since

$$x_1\,x_2\dots x_p=(-1)^{p-1}\phi(x),$$

and $f(x)$ is of the n^{th} degree, the coefficient must be $(-1)^{n(p-1)}$ which is the same as $(-1)^{p-1}$, n being a prime number.

An Example.

Suppose $\phi(x)\equiv x^2$, and $n=5$. From Theorem V we deduce that

$$(x^{31}-1)\equiv(x-1)\prod\{f(x)\}$$

where $f(x)$ is a cyclic expression of the fifth degree. We shall now proceed to examine the form of $f(x)$ in detail.

1. Consider the equation

$$x^5-p_1x^4+p_2x^3-p_3x^2+p_4x-p_5=0 \quad \dots \quad \dots \quad (\text{I})$$

whose roots are α, α^2, α^4, α^8, α^{16}. $(\alpha^{31}=1)$.

Put

$$\begin{aligned}
y_1&=\alpha+\alpha^2+\ldots\alpha^{16}\\
y_2&=\alpha^3+\alpha^6+\ldots\alpha^{48}(=\alpha^{17})\\
y_3&=\alpha^9+\alpha^{18}+\ldots\alpha^{20}\\
y_4&=\alpha^{27}+\alpha^{23}+\ldots\alpha^{29}\\
y_5&=\alpha^{19}+\alpha^7+\ldots\alpha^{25}\\
y_6&=\alpha^{26}+\alpha^{21}+\ldots\alpha^{13},
\end{aligned}$$

and $z_1=y_1+y_4$; $z_2=y_2+y$; $z_3=y_3+y_6$.

Then we easily find $\left.\begin{aligned}y_1y_2&=y_1+y_3+2y_5+y_6\\ y_1y_3&=y_2+y_3+y_5+2y_6\end{aligned}\right\}$

$$y_1y_4=z_1+z_2+5\,;\ y_2y_5=z_2+z_3+5\,;\ y_3y_6=z_3+z_1+5$$
$$z_1z_2=4z_1+3z_2+3z_3\,;\ \&c.\,;\ z_1^2=3z_1+4z_2+2z_3+10.$$

Thus
$$\left.\begin{aligned}\Sigma(z_1)&=\Sigma(y_1)=-1\\ \Sigma(z_1z_2)&=10\Sigma(z_1)=-10\\ z_1z_2z_3&=32\Sigma(z_1)+40=8\end{aligned}\right\},$$

so that z satisfies the equation

$$z^3+z^2-10z-8=0. \quad \ldots \quad \ldots \quad \text{(II)}$$

Hence y_1, y_4 are the roots of

$$y^2-z_1y+z_1+z_2+5=0\,;\ \&c.$$

Hence the equation whose roots are y_1, y_2, y_3, y_4, y_5, y_6 is

$$\mathrm{II}\ (y^2-z_1y+z_1+z_2+5)=0,$$

or
$$y^6+y^5+3y^8+11y^3+44y^2+36y+32=0, \quad \ldots \quad \ldots \quad \text{(III)}$$

by virtue of (II)

Also, from (I),

$$p_1=y_1\,;\ p_2=y_2+y_3\,;\ p_3=y_5+y_6\,;\ p_4=y_4\,;\ p_5=1.$$

Thus, finally, we have the following:

The cyclic equation

$$x^5-y_1x^4+(y_2+y_3)x^3-(y_5+y_6)x^2+y_4x-1=0$$

where y_1, $y_2 \ldots y_6$ are the roots of (III), has for roots α, α^2, α^4, α^8, α^{16}; and

$$\frac{x^{31}-1}{x-1}.\equiv\mathrm{II}\ \{\,x^5-y_1x^4+\ldots-1\,\}.$$

2. Equation (III) may be directly obtained as follows:—

Write down the values of y_1^2, y_1y_2, ……as linear expressions in terms of y_1, y_2, ………in a tabular form.

Thus we have

	y_1	y_2	y_3	y_4	y_5	y_6	
y_1^2	1	2	2	...	...	...	5
y_1y_2	1	...	1	...	2	1	
y_1y_3	...	1	1	...	1	2	
y_1y_4	1	1	...	1	1	...	
y_1y_5	1	1	...	2	1	...	
y_1y_6	...	1	...	2	1	1	

From the above table we get the values of y_1^3, y_1^4, y_1^5, y_1^6 at once as given below.

	y_1	y_2	y_3	y_4	y_5	y_6	
y_1^3	3	4	6	...	6	6	
y_1^4	13	24	16	24	26	22	
y_1^5	87	114	66	120	136	78	120
y_1^6	577	574	354	548	628	324	600

[Multiply the first three rows of the first table by 1, 2, 2 respectively and add; the result is y_1^3. Multiply the rows by 3, 4, 6, 0, 6, 6 in order and add to get y_1^4; and so on.]

From these we deduce

$$s_1=\Sigma(y_1)=-1\,; \quad s_2=\Sigma(y_1^2)=+5\ \Sigma(y_1)=-5\,;$$
$$s_3=\Sigma(y_1^3)=25\Sigma(y_1)=-25.$$
$$s_4=-125\,; \quad s_5=-601+720=119\,; \quad s_6=-3005+3600=595.$$

Hence by Newton's formulae

$$p_1=-1.$$
$$-2p_2=-5-1=-6. \quad \therefore\ p_2=3.$$
$$3p_3=-25-5-3=-33. \quad \therefore\ p_3=11.$$
$$-4p_4=-125-25-15-11=-176. \quad \therefore\ p_4=44.$$
$$5p_5=119-125-75-55-44=-180. \quad \therefore\ p_5=36.$$
$$-6p_6=595+119-375-275-220-36=-192. \quad \therefore\ p_6=32.$$

The equation in y is thus established, and is identical with (III).

3. The co-efficients in equation (III) may also be calculated by arranging the symmetric functions of *the* y's into groups.

Thus

$$\Sigma(y_1y_2)=(12)+(13)+\tfrac{1}{2}(14),$$

where (12) denotes $(y_1y_2+y_2y_3+y_3y_4+y_4y_5+y_5y_6+y_6y_1)$

(13) ,, $(y_1y_3+y_2y_4+y_3y_5+y_4y_6+y_5y_1+y_6y_2)$

(14) ,, $(y_1y_4+y_2y_5+y_3y_6+y_4y_1+y_5y_2+y_6y_1)$

Similarly

$$\Sigma(y_1y_2y_3)=(123)+(124)+(134)+\tfrac{1}{3}(135)$$
$$\Sigma(y_1y_2y_3y_4)=(1234)+(1235)+\tfrac{1}{2}(1245)$$
$$\Sigma(y_1y_2y_3y_4y_5)=(12345)$$

By means of the first table the several cyclic groups can be linearly expressed in terms of the *y's*, and the values of the symmetric functions found by substitution. The process, however, is tedious.

4. Returning to equation (I), write

$$y_1=\alpha+\alpha^{26}+\alpha^{25}+\alpha^{30}+\alpha^{5}+\alpha^{6}$$
$$y_2=\alpha^{3}+\alpha^{16}+\alpha^{13}+\alpha^{28}+\alpha^{15}+\alpha^{18}$$
$$y_3=\alpha^{9}+\alpha^{17}+\alpha^{8}+\alpha^{22}+\alpha^{14}+\alpha^{23}$$
$$y_4=\alpha^{27}+\alpha^{20}+\alpha^{24}+\alpha^{4}+\alpha^{11}+\alpha^{7}$$
$$y_5=\alpha^{19}+\alpha^{29}+\alpha^{20}+\alpha^{12}+\alpha^{2}+\alpha^{24}$$

following Gauss's method; we get

	y_1	y_2	y_3	y_4	y_5	
y^2	2	0	0	2	1	6
y_1y_2	0	1	2	1	2	
y_1y_3	0	2	2	1	1	
y_1y_4	2	1	1	0	2	
y_1y_5	1	2	1	2	0	

Proceeding as before, we find

y_1^3	15	4	3	6	6	12
y_1^4	60	28	26	49	38	90
y_1^5	346	205	195	250	240	360

Hence, we can at once write

$$s_1=p_1=-1.$$
$$s_2=\Sigma(y_1^2)=30-5=25.$$
$$s_3=\Sigma(y_1^3)=60-34=26.$$
$$s_4=\Sigma(y_1^4)=450-201=249.$$
$$s_5=\Sigma(y_1^5)=1800-1236=564.$$

And, from Newton's formulae, we deduce

$$p_2=-12,$$

$$3p_3=26+25+12=63. \quad \therefore\ p_3=21.$$

$$-4p_4=249+26-300+21=-4. \quad \therefore\ p_4=1.$$
$$5p_5=564+249-312-525-1=-125. \quad \therefore\ p_5=-25.$$

Hence the equation, whose roots are the y's, is

$$y^5+y^4-12y^3-21y^2+y+25=0. \qquad \dots \qquad \dots \qquad \text{(i)}$$

In other words,

$$(x^{31}-1/(x-1)\equiv\Pi\{x^6-y_1x^5+(y_1+y_4+3)x^4-(2y_1+y_5+2)x^3$$
$$+(y_4+y_1+3)x^2-y_1x+1\}.$$

where the y's are the roots of (i).

5. Lastly, the method of transformation may be employed to reach the same result. Thus, putting

$$f(x)\equiv x^5-p_1x^4+p_2x^3-p_3x^2+p_4x-p_5=0,$$

in § **1** above, we obtain the identity

$$f(x^2)\equiv-f(x).\,f(-x)\,;$$

whence, by equating co-efficients,

$$\left.\begin{array}{l} 2p_2=p_1^2-p_1\,;\ p_2^2-p_2=2(p_1p_3-p_4)\,; \\ 2p_3=p_4^2-p_4\,;\ p_3^2-p_3=2(p_4p_2-p_1)\,; \end{array}\right\}$$
$$p_5=1.$$

Also, giving x the value **1**, we have

$$p_1+p_2+p_3+p_4+1=0.$$

Write $p_1+p_4=z$, so that $p_2+p_3+z+1=0$; then eliminating p_2, p_3,

$$p_1^2+p_4^2+p_1+p_4+2=0,$$
$$z^2+z+2=2p_1p_4.$$

Again $p_4(p_2^2-p_3)-p_1(p_3^2-p_2)=2p_1p_4(p_3-p_2)-2(p_4^2-p_1^2)$, whence substituting for p_2, p_3 and dividing out by (p_1-p_4), we get

$$p_1p_4(p_1^2+p_1p_4+p_4^2)+2p_1p_4(p_1+p_4)-5p_1p_4-8(p_1+p_4)=0.$$

i.e.,
$$z^4+4z^3-7z^2-38z-24=0.$$

Rejecting the factor $(z+3)$* we are thus led to the cubic

$$z^3+z^2-10z-8=0,$$

which is the same a sequation (II).

The values of p_1, p_4 are readily found as the roots of the quadratic

$$y^2-zy+\tfrac{1}{2}(z^2+z+2)=0,$$

since $p_1+p_4=z$, and $p_1p_4=\frac{1}{2}(z^2+z+2)$.

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* The factor $(z+3)$ is rejected, because if $z=-3$, p_1 and p_4 are the roots of

$$p^2+3p+4=0\;;$$

and the quintic in this case becomes

$$x^5-px^4-2(p+1)x^3-2(p-2)x^2-(p+3)x-1=0,$$

which is the same as

$$(x^2+x+1)\{\,x^3-(p+1)x^2-(p+2)x-1\,\}=0.$$

The cubic factor here corresponds to the roots α, α^2, α^4 of the binomial equation $x^7-1=0$.

Mathematical Training from an Economic Point of View * (An Extract.)

Some one has said "a man's capacity for mathematics is paralleled by the development of the mathematical activity present in the great departments of investigation." Formerly the subject of biology was considered remotely distinct and separated from mathematical analysis. Then those students, who were not inclined to the mathematical discipline of their courses flocked to biologic studies only to learn that they had chosen intensely mathematical subjects.

It seems to me that our great work as mathematics teachers will be to lend our influence to the discovery of the relationships which underlie other fields of research. If these relationships belong to any mathematical category, as I believe they do, then we have a large problem whose solution shall not only conserve the mathematical field but shall add the greatest stimulus to mathematical training.

Let us determine in how far each of the arbitrary divisions of the subject matter of mathematics contributes to desirable human achievements. Let us give proportionate weights to these divisions which we designate as *algebra*, *geometry*, *analysis* and *mechanics*, and determine whether the time spent upon them is justified, or whether the raw product, the material of our text-books, is poor, or whether we are extravagant or indulgent in the luxuries of our subject. It is generally conceded that mathematics knows no poverty. If therefore we should discover that our arbitrary division of mathematics is wrong, let us have the courage to throw it out. We must recognize the cold hard fact that we are dealing with a commercial problem in which we shall be harder pressed. If we can give no adequate answer, then we are no better than the unskilled labourer in the street. The modern tendency in education is democratic, not aristocratic as men have vainly attempted to make it in the past. Our ranks are swollen with men and women whose ideals and culture are no higher than those of despised workers in other vocations. Professional life is no longer a guarantee of the homage of exaltation. Why is it so? Because we have introduced so much in our studies that is unreal. We have deceived ourselves into believing that our efficiency was measured by the number of students graduated or passed to a higher grade. What about all those young men and women who have been neglected because we thought their withdrawal from school or college was an expression of unfitness? Yes, an unfitness in most cases for an insufficient school system. Now as mathematics teachers how heavily have we played in the *role*. We must not be amazed then if we hear a challenge put on our cherished subjects. Will algebra, geometry, trigonometry, analytics, calculus, even arithmetic, stand the test when we ask without bias? Are they taught efficiently in their present text-book form?" They have served us well in the past, but had we not better tear them down and build new structures in their places? If we are content to let them remain for a longer time, we must repair and replenish them. Most of our teachers look upon algebra, geometry and trigonometry as if they were the body of materials embodied in a standard text-book on that subject. We have just as many blind teachers who regard the formula as wooden as if it were intended merely for substitution from data instead of for its larger purpose of interpretation. These impediments to efficiency in mathematical teaching are the result of our failure to recognize the world as a domain of relationships. It does seem timely that we should incorporate all these mathematical subjects and teach them more efficiently as an organized whole. We can teach mathematics more efficiently by presenting 'theory' through the problem and postponing the former until we need it and can make immediate use of it. In this way we can avoid a great amount of the drudgery of algebra.

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* From a paper by Mr. E. H. Koch Jr., of New York.

"As each subject grows it fringes" and in the garden of mathematics there are weeds to pull out. One of the most persistent weeds that grow among the flowers in our estate is the so-called "problem." Why do we permit these monstrosities, these unreal problems to fill our text-books? They represent a type of fiction which is revolting to the pupil. They are vicious in principle because they contradict the realities of life. They negate that highest virtue of mathematics, the search for truth. Every puzzle, or "faked" or "unreal" problem inserted in our text-books increases the denominator of the fraction or ratio of 'efficiency' and therefore decreases its value. We should establish the principle never to insert a problem which has to be worked backwards in contra-distinction to the realities of life. There is plenty of good material for real live problems, but if our knowledge of real things is too limited we had better omit the fictitious problems entirely.

It is a disgrace in this time and age to continue to suffer under the yoke of the authority of such text-books, and in the name of the thousands of students who are preparing for college and especially the technical school we protest against this infliction.

There has been reason for honest variance of opinion regarding the cultural and utilitarian value of mathematics as long as these claims, remained entrenched in the old undisputed doctrines: "There is no subject labelled cultural which is not at the same time practical and there is no subject labelled practical which is not at the same time cultural." Through mutual understanding we realize that education is beginning to teach subject-matter in terms of daily life and is recognizing the factors which are of value to our people. We have made the first serious attempt in the history of our education to meet the demand of our economic, industrial and social problems. We have learned that tremendous industrial forces have been developed without any commensurate adjustment of our school system. We may no longer disregard the significance of industry in its influence upon our life. Our systems and subject matter must be rehabilitated. Out of the prevailing systems there have emerged forms of practical mathematics which connect the school with the industries and vocations of our people. They involve a more or less special training which is best adapted to make the student discover for himself and to promote the qualities essential to his success in a specific vocation. We are getting a little closer to the various types of mind and the precise factors of intelligence.

Two years ago the speaker undertook the unification of the essentials of elementary mathematics for electrical students. The subject matter of this practical mathematics extends over the usual range of elementary mathematics including the notions and symbolism of the elements of calculus and vector analysis and vector applications. This was necessary in order to meet the students' requirements in both direct and alternating current work. All non-essentials are omitted and the teaching of theory is postponed until the student requires it in his work. We count this a great gain in economy. The student has a clear understanding of his work. He can express ideas intelligently. He has the power of applying knowledge practically. He realizes the right thing to do in an emergency. He accomplishes a reasonable amount of work in a definite time. His class room is always a work room. His work is interesting. It gives him pleasure. He has greater physical endurance, poise, displays greater fidelity, assurance, neatness, accuracy and the greatest of all blessings 'common-sense.' He is also trained for leadership and executive positions by imposing upon him responsibility of foremanship work in the class room, laboratory and shop. Upon entering the class room he immediately reports his outsigned work on a time sheet on the front cover of his work-book and makes a similar entry at the close of the period. The work-book contains every scrap of work which the student has completed and this book is retained by the instructor who has the student's entire work under constant supervision and inspection. The work of the instructor is made more efficient by the co-operation of the students through their foreman. We are simply recognizing a well established principle that a man can learn to direct others by first learning to be guided by others. Secondly, he can learn to guide others best by working with them, sharing their trials and disappointments, and gaining an intimate appreciation of and sympathy with their modes of thought and action. "Of the many evils which our examination system has inflicted upon us the central one has consisted in forcing our school and University teaching into moulds determined not by the true interests of education but the mechanical exigencies of the examination syllabus." †

The time is not far distant when our college and State examination Boards will recognize these facts and they will be compelled to offer examinations in a unified course such as practical mathematics. These demands will come first from the technical schools.

We cannot all be great mathematicians, and so our universities would become more efficient if they did not attempt to duplicate work which could be undertaken better by a few institutions. I wish we would hear the phrase 'institutions for teaching' very more often than we hear about 'institutions for learning.' Would it not be the part of wisdom if many of our institutions devoted their time in discovering how mathematics is contributing to other fields of intellectual activities and thereby promote such projects? We have seen an increased efficiency in mathematics teaching since history stepped in and opened a new pathway. We should establish experiment stations not only in agriculture and engineering, but also in mathematics and other subjects as well. Then the study of higher mathematics will have an immediate justification as well as blazing the future pathway of knowledge.

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† Science, September 23, 1910.

SHORT NOTES.

On the Integral $\int_0^x \frac{\tan^{-1}t}{t}dt$.

1. Let

$$\phi(x)=\int_0^x \frac{\tan^{-1}t}{t}dt. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Then it is easy to see that

$$\phi(x)+\phi(-x)=0\,; \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and that

$$\phi(x)=\frac{x}{1^2}-\frac{x^3}{3^2}+\frac{x^5}{5^2}-\frac{x^7}{7^2}+\ldots \quad \ldots \quad \ldots \quad (3)$$

provided that $|x|\leq 1$.

Changing t into $1/t$ in (1), we obtain

$$\phi(x)-\phi\left(\frac{1}{x}\right)=\frac{1}{2}\pi \log x, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

provided that the real part of x is positive.

The results in the following two sections can be very easily proved by differentiating both sides with respect to x.

2. If $0<x>\frac{1}{2}\pi$, then

$$\frac{\sin 2x}{1^2}+\frac{\sin 6x}{3^2}+\frac{\sin 10x}{5^2}+ \quad \ldots \quad \ldots \quad (5)$$

$$=\phi(\tan x)-x\log(\tan x).$$

If, in particular, we put $x=\frac{1}{8}\pi$ and $\frac{1}{12}\pi$ in (5), we obtain

$$\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\ldots=\frac{1}{\sqrt{2}}\phi(\sqrt{2}-1)+\frac{\pi}{8\sqrt{2}}\log(1+\sqrt{2})+\frac{\pi^2}{16}\,; \quad \ldots \quad (6)$$

and

$$2\phi(1)=3\phi(2-\sqrt{3})+\frac{1}{4}\pi\log(2+\sqrt{3}). \quad \ldots \quad \ldots \quad (7)$$

If $-\frac{\pi}{2}<x<\pi$, then

$$2\phi\left(\tan\frac{x}{2}\right)=\sin x+\frac{2}{3}.\ \frac{\sin^3 x}{3}+\frac{2\cdot4}{3\cdot5}.\ \frac{\sin^5 x}{5}+ \quad \ldots \quad (8)$$

If $0<x<\frac{1}{2}\pi$, then

$$\frac{\sin x}{1^2}\cos x+\frac{\sin 2x}{2^2}\cos^2 x+\frac{\sin 3x}{3^2}\cos^3 x+\ldots$$

$$=\phi(\tan x)+\tfrac{1}{2}\pi\log\cos x-x\log\sin x\,;\ldots \quad (9)$$

and

$$\frac{\cos x+\sin x}{1^2}+\frac{1}{2}.\ \frac{\cos^3 x+\sin^3 x}{3^2}+\frac{1\cdot3}{2\cdot4}.\ \frac{\cos^5 x+\sin^5 x}{5^2}+\ldots$$

$$=\phi(\tan x)+\tfrac{1}{2}\pi\log(2\cos x). \quad (10)$$

If $-\frac{1}{2}\pi<x<\frac{1}{2}\pi$ and a be any number such that

$\left|(1-a)\sin x)\right|\leq 1$, and $\left|\left(1-\frac{1}{a}\right)\cos x\right|\leq 1$, then

$$\frac{\sin x}{1^2}\left(1-\frac{1}{a}\right)\cos x+\frac{\sin 2x}{2^2}\left(1-\frac{1}{a}\right)^2\cos^2 x+\frac{\sin 3x}{3^2}\left(1-\frac{1}{a}\right)^3\cos^3 x+\ldots$$
$$+\frac{\sin(x+\frac{1}{2}\pi)}{1^2}(1-a)\sin x+\frac{\sin 2(x+\frac{1}{2}\pi)}{2^2}(1-a)^2\sin^2 x+\ldots$$
$$=\phi(\tan x)-\phi(a\tan x)+x\log a. \quad \ldots \quad (11)$$

3. Let $\mathrm{R}(x)$ and $\mathrm{I}(x)$ denote the real and the imaginary parts of x respectively. Then, if $-1<\mathrm{R}(x)<1$,

$$\log\left(1-\frac{x^2}{1^2}\right)-3\log\left(1-\frac{x^2}{3^2}\right)+5\log\left(1-\frac{x^2}{5^2}\right)-\ldots\ldots$$
$$=\frac{4}{\pi}\left[\phi(1)-\phi\left\{\tan\tfrac{1}{4}\pi(1-x)\right\}\right]+\log\tan\tfrac{1}{4}\pi(1-x). \quad (12)$$

Putting $x=\frac{2}{3}$ in (12) and using (7), we obtain

$$\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{9^2}\right)^{-3}\left(1-\frac{4}{15^2}\right)^{5}\left(1-\frac{4}{21^2}\right)^{-7}\left(1-\frac{4}{27^2}\right)^{9}\ldots$$
$$=(2-\sqrt{3})^{\frac{2}{3}}e^{n}, \text{ where } n=\frac{4}{3\pi}\phi(1). \quad (13)$$

Again subtracting $\log(1-x)$ from both sides in (12) and making $x\to 1$, we obtain

$$\left(1-\frac{1}{3^2}\right)^{-3}\left(1-\frac{1}{5^2}\right)^{5}\left(1-\frac{1}{7^2}\right)^{-7}\left(1-\frac{1}{9^2}\right)^{9}\ldots\ldots$$
$$=\frac{\pi}{8}e^{\frac{4}{\pi}\phi(1)}. \quad (14)$$

If $-1<\mathrm{I}(x)<1$, then

$$\log\left(1+\frac{x^2}{1^2}\right)-3\log\left(1+\frac{x^2}{3^2}\right)+5\log\left(1+\frac{x^2}{5^2}\right)-\ldots$$
$$=\frac{4}{\pi}\left\{\phi(1)-\phi\left(e^{-\frac{1}{2}\pi x}\right)\right\}2x\tan^{-1}e^{-\frac{1}{2}\pi x}. \quad (15)$$

From this and (7), we see that if $\frac{1}{2}\pi x=\log(2+\sqrt{3})$, then

$$\left(1+\frac{x^2}{1^2}\right)^{1}\left(1+\frac{x^2}{3^2}\right)^{-3}\left(1+\frac{x^2}{5^2}\right)^{5}\left(1+\frac{x^2}{7^2}\right)^{-7}\ldots\ldots=e,\frac{4}{3\pi}\phi(1), \quad (16)$$

where n is the same as in (13).

It follows at once from (12) and 15 that, if

$-1<\mathrm{R}(\beta)<1, -1<\mathrm{I}(\alpha)<1,$

then

$$e^{\frac{1}{2}\pi\alpha\beta}=\left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)^{1}\left(\frac{3^2-\beta^2}{+\alpha^2}\right)^{3}\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^{5}\left(\frac{7^2-\beta^2}{7^2+\alpha^2}\right)^{7}\ldots \quad (17)$$

provided that $\cosh\frac{1}{2}\pi\alpha=\sec\frac{1}{2}\pi\beta$.

4. Now changing x into $2x(1+i)$ in (15), we have

$$\log\left(1+\frac{8ix^2}{1^2}\right)-3\log\left(1+\frac{8ix^2}{3^2}\right)+5\log\left(1+\frac{8ix^2}{5^2}\right)-\ldots\ldots$$
$$=\frac{4}{\pi}\phi(1)-4x(1+i)\ \tan^{-1}e^{-\pi x(1+i)}$$
$$-\frac{4}{\pi}\left\{\frac{1}{1^2}e^{-\pi x(1+i)}-\frac{1}{3^2}e^{-\pi x(1+i)}+\ldots\right\}$$

Equating real and imaginary parts, we see that if x is positive, then

$$\log\left(1+\frac{64x^4}{1^4}\right)-3\log\left(1+\frac{64x^4}{3^4}\right)+5\log\left(\frac{64x^4}{1+5^4}\right)-\ldots$$
$$=\frac{8}{\pi}\phi(1)-2x\ \log\left(\frac{\cosh\pi x+\sin\pi x}{\cosh\pi x-\sin\pi x}\right)-4x\ \tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$
$$-\frac{8}{\pi}\left\{\frac{\cos\pi x}{1^2}e^{-\pi x}-\frac{\cos 3\pi x}{3^2}e^{-3\pi x}+\frac{\cos 5\pi x}{5^2}e^{-5\pi x}-\ldots\right\};\quad(18)$$

and

$$\tan^{-1}\frac{8x^2}{1^2}-3\ \tan^{-1}\frac{8x^2}{3^2}+5\ \tan^{-1}\frac{8x^2}{5^2}-$$
$$=x\log\left(\frac{\cosh\pi x+\sin\pi x}{\cosh\pi x-\sin\pi x}\right)-2x\ \tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$
$$+\frac{4}{\pi}\left\{\frac{\sin\pi x}{1^2}e^{-\pi x}-\frac{\sin 3\pi x}{3^2}e^{-3\pi x}+\frac{\sin 5\pi x}{5^2}e^{-5\pi x}-\ldots\right\}\quad(19)$$

It follows from (18) that, if n is a positive odd integer, then

$$\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\ldots\ldots$$
$$=e^{\frac{8}{\pi}\phi(1)}\left(\frac{1-e^{-\frac{1}{2}\pi n}}{1+e^{-\frac{1}{2}\pi n}}\right)^{2n\cos\frac{1}{2}(n-1)\pi}\quad(20)$$

and if n is any even integer then

$$\left(1+\frac{4n^4}{1^4}\right)\left(1+\frac{4n^4}{3^4}\right)^{-3}\left(1+\frac{4n^4}{5^4}\right)^{5}\left(1+\frac{4n^4}{7^4}\right)^{-7}\ldots\ldots\quad\ldots\quad(21)$$
$$=\exp\left\{\frac{8}{\pi}\phi(1)-\frac{8}{\pi}(-1)^{\frac{1}{2}n}\left(\phi\left(e^{-\frac{1}{2}\pi a}\right)+\frac{1}{2}\pi n\ \tan^{-1}e^{-\frac{1}{2}\pi n}\right)\right\}$$

Similarly from (19) we see that, if n is any positive odd integer then

$$\tan^{-1}\frac{2n^2}{1^2}-3\ \tan^{-1}\frac{2n^2}{3^2}+5\ \tan^{-1}\frac{2n^2}{5^2}-\ldots\ldots$$
$$=\frac{4}{\pi}(-1)^{\frac{1}{2}(n-1)}\left\{\frac{\pi n}{4}\log\left(\frac{1+e^{-\frac{1}{2}\pi n}}{1-e^{-\frac{1}{2}\pi n}}\right)+\frac{1}{1^2}e^{-\frac{1}{2}\pi n}\right.$$
$$\left.+\frac{1}{3^2}e^{-\frac{3}{2}\pi n}+\frac{1}{5^2}e^{-\frac{5}{2}\pi n}+\ldots\ldots\right\};\quad(22)$$

and, if n is a positive even integer, then

$$\tan^{-1}\frac{2n^2}{1^2}-3\tan^{-1}\frac{2n^2}{3^2}+5\tan^{-1}\frac{2n^2}{5^2}-\ldots\ldots$$

$$=2n(-1)^{\frac{1}{2}n-1}\tan^{-1}e^{-\frac{1}{2}\pi n}. \qquad (23)$$

In this connection it may be interesting to note that

$$\tan^{-1}e^{-\frac{1}{2}\pi n}=\frac{\pi}{4}-\left(\tan^{-1}\frac{n}{1}-\tan^{-1}\frac{n}{3}+\tan^{-1}\frac{n}{5}-\ldots\ldots\right\} \quad \ldots \quad (24)$$

for all real values of n.

5. Remembering that

$$\frac{\pi}{4\cosh\pi x}=\frac{1}{1^2+4x^2}-\frac{3}{3^2+4x^4}+\frac{5}{5^2+4x^2}-\ldots\ldots$$

we have

$$\frac{\pi}{4}\sum_{1}^{\infty}\frac{1}{n^2\cosh\pi n x}=\sum_{n=1}^{n=\infty}\left\{\frac{1}{n^2(1^2+4n^2x^2)}-\frac{3}{n^2(3^2+4n^2x^2)}+\ldots\ldots\right\}$$

$$=\frac{\pi^3}{16}\left(\frac{1}{3}+\frac{x^2}{2}\right)-\frac{\pi x}{2}\left(\frac{\coth\frac{\pi}{2x}}{1^2}-\frac{\coth\frac{3\pi}{2x}}{3^2}+\frac{\coth\frac{5\pi}{2x}}{5^2}-\ldots\ldots\right) \qquad (25)$$

That is to say, if α and β are real and $\alpha\beta=\pi^2$, then

$$\phi(1)+2\phi(e^{-\alpha})+2\phi(e^{-2\alpha})+2\phi(e^{-3\alpha})+\ldots \quad \ldots$$

$$=\frac{\pi}{8}\left(\frac{\alpha}{3}+\frac{\beta}{2}\right)-\frac{\pi}{4\beta}\left\{\frac{1}{1^2\cosh\beta}+\frac{1}{2^2\cosh 2\beta}+\ldots\right\} \qquad (26)$$

If, in particular, we put $\alpha=\beta=\pi$ in (26), we obtain

$$\phi(1)=\frac{5\pi^2}{48}-2\left\{\frac{1}{1^2(e^{\pi}-1)}-\frac{1}{3^2(e^{-3}\pi-1)}+\frac{1}{5^2(e^{5}\pi-1)}\ldots\right\}$$

$$-\frac{1}{2}\left\{\frac{1}{1^2(e^{\pi}+e^{-\pi})}+\frac{1}{2^2(e^{2\pi}+e^{-2\pi})}+\frac{1}{3^2(e^{3\pi}+e^{-3\pi})}+\ldots\right\} \qquad (27)$$

$$=\cdot 9\,1\,5\,9\,6\,5\,5\,9\,4\,2,$$

approximately.

S. RAMANUJAN.

A Theorem on Areas.

1. Let a number of particles $a_1, a_2 \ldots a_n$ of masses $m_1, m_2 \ldots m_n$ move in any manner in a plane and return to their initial positions *simultaneously*, after describing curves of areas $A_1, A_2 \ldots A_n$. Then the area G described by their centroid g is given by

$$\Sigma(m_1).G=\Sigma(m_1 A_1)-\Sigma(m_1 R_1),$$

where R_1, R_2, ... are the areas enclosed by the paths of a_1, a_2, ... relative to g.

For, we have

$$A=\int(y\,dx),\ G=\int(\bar{y}\ d\bar{x}),\ R=\int\{\,(y-\bar{y})\,d(x-\bar{x})\,\}\ ;$$

$$\bar{x}=\Sigma(mx)/\Sigma(m),\ \bar{y}=\Sigma(my)/\Sigma(m),\ d\bar{x}=\Sigma(m\,dx)/\Sigma(m),\ \&c.$$

$$\begin{aligned}\therefore\ \Sigma(mR)&=\Sigma\int\{\,m(y-\bar{y})\,d(x-\bar{x})\,\}\\ &=\Sigma(m\int y\,dx)+\Sigma(m\int\bar{y}\,d\bar{x})-\Sigma(m\int y\,d\bar{x})-\Sigma(m\int\bar{y}\,dx)\\ &=\Sigma(mA)+\Sigma(mG)-\Sigma(\int my\,d\bar{x})-\int\bar{y}\Sigma(m\,dx)\\ &=\Sigma(mA)+\Sigma(m).G-\Sigma(m).G-\Sigma(m).G\\ &=\Sigma(mA)-\Sigma(m).G.\end{aligned}$$

Hence $$\Sigma(m).\,G=\Sigma(mA)-\Sigma(mR).$$

Cor. 1. In particular, if the particles move as a rigid system, the relative paths are all circles; we therefore have

$$\Sigma(m).G=\Sigma(mA)-\pi.I$$

where I is the moment of inertia of the system about g.

Cor. 2. If the particles are free to move in a line, while the line itself moves in any manner parallel to itself, the relative paths are segments of straight lines and $R=0$. The formula becomes

$$\Sigma(m).G=\Sigma(m.A).$$

2. Consider three particles a_1, a_2, a_3, moving as a rigid system and describing equal areas A. Here

$$G=A-\pi.k^2$$

where k=radius of gyration of the particles round g.

Also, if r be the circumradius of the invariable triangle $a_1\,a_2\,a_3$, and o the circumcentre

$$k^2=r^2-og^2,$$

whence

$$G=A-\pi r^2+\pi og^2.$$

Thus, all points g in the plane of $a_1\,a_2\,a_3$ which describe equal areas (as a_1, a_2, a_3 describe equal areas A) lie on a circle with centre o; the area described by o is $(A-\pi r^2)$ and is the minimum area for points in the plane.

Cor. 1. If x, y, z, be the areal co-ordinates of g with respect to $a_1\,a_2\,a_3$, the above result can be written

$$G=xA_1+yA_2+zA_3+\pi t^2,$$

where t is the tangent from g to the circumcircle.

Cor. 2. On any straight line there is one point for which the area described is less than for any other point: this point being the foot of the perpendicular from o on the line.

If p, q, r be three collinear points, then

$$qr.P+rp.Q+pq.R+\pi.pq.qr.rp=0,$$

since $P=A-\pi r^2+\pi.op^2$, &c.

3. In the previous article, let circles be described round each point of the plane equal in area to the area described by the point. Then, *the radical axis of any two circles of the system passes through the circumcentre* o.

For, if p_1, p_2 be two points and ρ_1, ρ_2 the radii of the corresponding circles, we have

$$P_1=\pi\ \rho_1^2=A-\pi\ r^2+\pi op_1^2,$$
$$P_2=\pi\ \rho_2^2=A-\pi\ r^2+\pi.op_2^2$$
$$\therefore \qquad op_1^2-op_2^2=\rho_1^2-\rho_2^2.$$

Cor. Any three circles drawn as above have the minimum point o for their radical centre; and all the circles have a common orthogonal circle.

R. Vythynathaswamy.

SOLUTIONS.

Question 358.

(S. RAMANUJAN) :—If n is a multiple of 4 excluding zero, shew that

$$1^{n-1}\operatorname{sech}\frac{\pi}{2}-3^{n-1}\operatorname{sech}\frac{3\pi}{2}+5^{n-1}\operatorname{sech}-+\ldots=0.$$

Solution by M. Bhimasena Rao.

We know

$$\int_0^\infty \frac{2\cos 2xt}{\cosh \pi t}\,dt=\operatorname{sech} x.$$

$$\therefore \int_0^\infty \frac{2\sin xt.\sin at}{\cosh \pi t}\,dt=\int_0^\infty \frac{\cos(x-a)t-\cos(x+a)t}{\cosh \pi t}\,dt$$

$$=\tfrac{1}{2}\left(\operatorname{sech}\frac{x-a}{2}-\operatorname{sech}\frac{x+a}{2}\right).$$

Also,

$$\int_0^\infty \frac{2\sin at}{\cosh \pi t}.\frac{\sin(4n+2)xt}{2\cos xt}\,dt$$

$$=\int_0^\infty \frac{2\sin at.dt}{\cosh \pi t}\,(\sin xt-\sin 3xt+\sin 5xt\ldots\ldots\ldots+\sin(4n+1)xt).$$

$$=\tfrac{1}{2}\left\{\operatorname{sech}\frac{x-a}{2}-\operatorname{sech}\frac{3x-a}{2}+\operatorname{sech}\frac{5x-a}{2}\ldots\ldots\ldots\right\}$$

$$-\tfrac{1}{2}\left\{\operatorname{sech}\frac{x+a}{2}-\operatorname{sech}\frac{3x+a}{2}+\operatorname{sech}\frac{5x+a}{2}\ldots\ldots\ldots\right\}\ \ldots\quad(1)$$

Now

$$\int_0^\infty \frac{2\sin at}{\cosh \pi t}.\frac{\sin(4n+2)xt}{2\cos xt}\,dt$$

$$=\int_0^\infty\left\{\frac{\sin\frac{at}{2x}}{\cosh\frac{\pi t}{2x}}.\frac{\sin(2n+1)t}{2\cos\frac{t}{2}}\frac{dt}{x}\right\},\text{ by changing }2xt\text{ into }t$$

$$=\int_0^\infty\left\{\frac{\sin\frac{at}{2x}.\sin\frac{t}{2}}{\cosh\frac{\pi t}{2x}}.\frac{\sin(2n+1)t}{\sin t}\frac{dt}{x}.\right\}$$

Let n become infinite; then the value of the above integral is

$$\frac{\pi}{x}\left\{\frac{\sin\frac{a\pi}{2x}}{\cosh\frac{\pi^2}{2x}}-\frac{\sin\frac{3a\pi}{2x}}{\cosh\frac{3\pi^2}{2x}}+\frac{\sin\frac{5a\pi}{2x}}{\frac{5\pi^2}{2x}}\ldots\ldots\right\}$$

$$\tfrac{1}{2}\left\{\operatorname{sech}\frac{x-a}{2}-\operatorname{sech}\frac{3x-a}{2}+\operatorname{sech}\frac{5x-a}{2}\dots\dots\text{ to }\infty\right\}$$

$$=e^{-\frac{1}{2}(x-a)}-e^{-\frac{3}{2}(x-a)}+e^{-\frac{5}{2}(x-a)}-\dots\dots\dots$$

$$-e^{-\frac{1}{2}(3x-a)}+e^{-\frac{3}{2}(3x-a)}-e^{-\frac{5}{2}(3x-a)}+\dots\dots\dots$$

$$\begin{matrix}\dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\end{matrix}$$

This double series is absolutely convergent if $x>a>o$. Summing by columns, we get

$$\frac{1}{2}\left\{\frac{e^{\frac{a}{2}}}{\cosh\frac{x}{2}}-\frac{e^{\frac{3a}{2}}}{\cosh\frac{3x}{2}}+\frac{e^{\frac{5a}{2}}}{\cosh\frac{5x}{2}}\dots\dots\dots\dots\right\}$$

Similarly,

$$\frac{1}{2}\left\{\operatorname{sech}\frac{x+a}{2}-\operatorname{sech}\frac{3x+a}{2}+\operatorname{sech}\frac{5x+a}{2}\dots\dots\dots\right.$$

$$=\frac{1}{2}\left\{\frac{e^{-\frac{a}{2}}}{\cosh\frac{x}{2}}-\frac{e^{-\frac{3a}{2}}}{\cosh\frac{3x}{2}}+\frac{e^{-\frac{5a}{2}}}{\cosh\frac{5x}{2}}\dots\dots\dots\dots\right\}$$

$\therefore$ Thus the right hand side of (1) when n is infinite becomes

$$\frac{\sinh\frac{a}{2}}{\cosh\frac{x}{2}}-\frac{\sinh\frac{3a}{2}}{\cosh\frac{3x}{2}}+\frac{\sinh\frac{5a}{2}}{\cosh\frac{5x}{2}}\dots\dots\dots$$

Hence $$\sum_{n=0}^{\infty}(-1)^n\frac{\pi}{x}\sin\frac{a\pi(2n+1)}{2x}.\ \operatorname{sech}\frac{\pi^2(2n+1)}{2x}$$

$$=\sum_{n=0}^{\infty}(-1)^n\sinh\frac{a(2n+1)}{2}.\ \operatorname{sech}\frac{x(2n+1)}{2}\qquad\dots\quad(2)$$

Equationg the coefficients of a^{2m+1}, we get

$$\sum_{n=0}^{\infty}(-1)^n\left[(2n+1)^{2m+1}\left\{\operatorname{sech}\frac{x(2n+1)}{2}-\right.\right.$$

$$\left.\left.(-1)^m\left(\frac{\pi}{x}\right)^{2m+2}\operatorname{sech}\frac{\pi^2(2n+1)}{2x}\right\}\right]=0.\qquad\dots\quad(3)$$

Putting $x=\pi$, we have

$$\sum_{n=0}^{\infty}(-1)^n\left[(2n+1)^{2m+1}\left\{\operatorname{sech}\frac{\pi(2n+1)}{2}\left(1-(-1)^m\right)\right\}\right]=0.$$

When m is even we get no effective result, as the left hand side vanishes. When $m=2r-1$, $(r \neq 0)$

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)^{4r-1}\operatorname{sech}\frac{\pi(2n+1)}{2}=0,$$

which is the required result with r instead of n.

When r is zero, the series becomes

$$\operatorname{sech}\frac{\pi}{2}-\operatorname{sech}\frac{3\pi}{2}+\frac{1}{5}\operatorname{sech}\frac{5\pi}{2}\ldots\ldots\ldots\ldots$$

The value of this series is $\frac{\pi}{8}$ and is got by making $n=1$ in Mr. S. Ramanujan's series of Question 353 or by Question 626 (vi).

Question 418.

(M. Bhimasena Rao) :—Tangents at P, Q, R, S on a cubic curve having a centre, co-intersect on the curve. Show that the mean centre of P, Q, R, S, is the centre of the curve, each asymptote bisecting a pair of opposite connectors of the quadrangle P Q R S.

Solution by N. Sankara Aiyar, M.A.

Taking the origin at the centre and two of the asymptotes as axes, since the curve has a centre, its equation can be reduced to the form

$$(lx+my)xy+ax+by=0.$$

The tangent at (f, g) on this is

$$(2lgf+mg^2+a)\xi+(2mgf+lf^2+b)\eta+2af+2bg=0.$$

If the point (ξ, η) be common to this tangent and the curve, by eliminating g between this equation and $(lf+mg)fg+af+bg=0$, we shall get an equation giving values of f for which the tangents will be concurrent at (ξ, η).

Now $\quad g^2m\xi+g(2lf\xi+2mf\eta+2b)+a\xi+lf^2\eta+b\eta+2af=0$

and

$$g^2fm+g(lf^2+b)+af=0.$$

$$\therefore \{fm(a\xi+lf^2\eta+b\eta+2af)-afm\xi\}^2$$
$$=\{mf(lf^2+b)-fm(2lf\xi+2mf\eta+2b)\}\times$$
$$\{af(2lf\xi+2mf\eta+2b)-lf^2+b)(a\xi+lf^2\eta+b\eta+2af)\}$$

Evidently two of the roots of this sextic are f.

To find the equation giving the four required roots, we divide this by $f^2-2\xi f+\xi^2$.

Now the coefficient of f^6 is

$$l^3\xi\eta+2ml^2\eta^2-ml^2\eta^2=l^2\eta(l\xi+m\eta).$$

The coefficient of f^5 is

$$(2l^2\eta b+2al^2\xi-2alm\eta+2alm\eta)=2l^2(a\xi+b\eta)$$

$$=-2l^2\xi\eta(l\xi+m\eta), \text{ by the equation of curve.}$$

The coefficient of f is

$$2b^2(a\xi+b\eta)=-2b^2\xi\eta(l\xi+m\eta)$$

The absolute term is

$$-b^2\xi(a\xi+b\eta)=b^2\xi^2\eta(l\xi+m\eta)$$

Hence the equation reduces to

$$(f^2-2\xi f+\xi^2)\left(f^4+pf^2+\frac{b^2}{l^2}\right)=0.$$

i.e. $f^4+pf^2+\dfrac{b^2}{l^2}=0$ is the equation giving P, Q, R, S.

i.e. $(f^2-\alpha^2)(f^2-\beta^2)=0.$

i.e. $f=\pm\alpha, \pm\beta.$

Hence the ξ—co-ordinate of the mean centre of P, Q, R, S; is $\dfrac{\alpha-\alpha+\beta-\beta}{4}=0$. Similarly the η—co-ordinate is 0. Hence the centre is the mean centre of P, Q, R, S.

Again, the middle point of PQ will have co-ordinates $\dfrac{\alpha-\alpha}{2}=0$. Hence this point lies on one of the asymptotes; similarly that of R S lies on this. Similarly the other two pairs of opposite connectors are bisected by the other two asymptotes.

Question 468.

(N. Sankara Aiyar, M.A.):—Solve the equation

$$\int_0^{v_0}\left\{\frac{x^2\left(\dfrac{dy}{dx}\right)^2+1}{v_0^2-x^2}\right\}^{\frac{1}{2}}dx=k.$$

Solution by A. Narasinga Rao.

Put $x=v_0\cos\theta$ ... ... (1)

The integral equation reduces to

$$\int_0^{\frac{\pi}{2}}\left\{\cot^2\theta\left(\frac{dy}{d\theta}\right)^2+1\right\}^{\frac{1}{2}}d\theta=k;$$

i.e. $$\int_{0}^{\frac{\pi}{2}} f(\theta)d\theta=k,$$

i.e. $$F\left(\frac{\pi}{2}\right)-F(o)=k.$$

where $F(\theta)=\int f(\theta)d\theta$ ... ... (2)

If any solution F of (2) be obtained, we have a corresponding f and y is given in terms of θ by the equation.

$$\cot^2\theta\left(\frac{dy}{d\theta}\right)^2+1=\{f(\theta)\}^2.$$

The elimination of θ between this value of y and (1) determines y as a function of x.

Put $F(\epsilon)=k\sin\theta+\theta\left(\theta-\frac{\pi}{2}\right)G(\theta)+\alpha$, α being an arbitrary constant. This value of F will satisfy (2) if G is an one-valued function of ϵ, finite and continuous in the range $0\leq\theta\leq\frac{\pi}{2}$.

For instance, take the simplest case $\alpha=0$, $G(\theta)=0$.

The relation determining y is now

$$\cot^2\theta\left(\frac{dy}{d\theta}\right)^2+1=k^2\cos^2\theta,$$

i.e. $$x^2\left(\frac{dy}{dx}\right)^2+1=k^2\frac{x^2}{v_o^2}\text{ from(1)}$$

Hence $$y=\int\left(\frac{k^2}{v_o^2}e^{2z}-1\right)^{\frac{1}{2}}dz,$$

where $z=\log x$.

Other forms of F may be obtained by inspection from equation (2). For instance

(i) $F(\theta)=\Sigma(A_r\sin^r\theta+B_r\cos^r\theta)$ the coefficients being connected by the relation $\sum_{r=1}(A_r-B_r)=k$

(ii) $F(\theta)=a^{2\theta/\pi}G(\sin\overline{4\theta+c})$ where G is one-valued and a, c are connected by $(a-1)G(\sin c)=k$

Questions 489, 608, 626.

Q. 489. (S. Ramanujan) :—Shew that

$$(1+e^{-\pi\sqrt{55}})(1+e^{-3\pi\sqrt{55}})(1+e^{-5\pi\sqrt{55}})\ldots\ldots$$
$$=\frac{1+\sqrt{(3+2\sqrt{5})}}{\sqrt{2}}\left\{e^{-\pi\sqrt{55}}\right\}^{\frac{1}{24}}$$

Q. 608. (M. BHIMASENA RAO) :—Shew that

$$\text{(i)}\quad (1+e^{-\pi})(1+e^{-3\pi})(1+e^{-5\pi})\ldots=2^{\frac{1}{4}}e^{-\frac{\pi}{24}}$$

$$\text{(ii)}\quad (1+e^{-\pi\sqrt{3}})(1+e^{-3\pi\sqrt{3}})\ldots=2^{\frac{1}{3}}\cdot e^{-\frac{\pi\sqrt{3}}{24}}$$

$$\text{(iii)}\quad (1+e^{-\pi\sqrt{7}})(1+e^{-3\pi\sqrt{7}})\ldots=2^{\frac{1}{2}}\cdot e^{-\frac{\pi\sqrt{7}}{24}}.$$

Q. 626. (M. BHIMASENA RAO) :—If $f(x)$ denotes

$$\operatorname{sech}\frac{\pi x}{2}-\frac{1}{3}\operatorname{sech}\frac{3\pi x}{2}+\frac{1}{5}\operatorname{sech}\frac{5\pi x}{2}-\ldots$$

shew that

$$\text{(i)}\ f(1)=\frac{\pi}{8}$$

$$\text{(ii)}\ f(\sqrt{3})=\frac{\pi}{24}$$

$$\text{(iii)}\ f(\sqrt{7})=\frac{1}{2}\left[\sin^{-1}\left(\frac{3}{4}\right)-\frac{\pi}{4}\right]$$

$$\text{(iv)}\ f(\sqrt{15})=\frac{1}{2}\left(\frac{\pi}{12}-\,-\sin^{-1}\frac{1}{4}\right).$$

$$\text{(v)}\ f\left(\sqrt{\frac{3}{5}}\right)=\frac{1}{2}\left(\cos^{-1}\frac{1}{4}-\frac{\pi}{12}\right)$$

$$\text{(vi)}\ f(x)+f(x^{-1})=\frac{\pi}{4}.$$

Are there any values of x other than zero and those given by (vi) for which $f(x)$ can be evalatted in finite terms?

Solution by A. C. L. Wilkinson, M.A., F.R.A.S.

Weber, Lehr buct der Algebra, Vol III, P. 86 **(11)** gives

$$q^{-\frac{1}{24}}\prod_{1}^{\infty}(1+q^{2n-1})=f(w),\quad\text{where } q=e^{i\pi w}$$

Also in the table of class invariants

$$f(\sqrt{-55})=\sqrt{2}\bar{x},\ n^2-x=\frac{1+\sqrt{5}}{2},$$

whence Mr. Ramanujan's result follows.

Again $f(\sqrt{-1})=\sqrt[4]{2}$, $f(\sqrt{-3})=\sqrt[3]{2}$, $f(\sqrt{-7})=\sqrt{2}$, whence the results of Question 608.

Jacobi, Fundamenta Nova, p. 108 (46) gives the formula

$$\frac{1}{2}\sin^{-1}k=\frac{2}{q^{1/2}+q^{-\frac{1}{2}}}-\frac{1}{3}\cdot\frac{2}{q^{3/2}+q^{-\frac{3}{2}}}+\frac{1}{5}\cdot\frac{2}{q^{5/2}+q^{-\frac{5}{2}}}-\ldots$$

Put $$q=e^{-\pi x}=e^{-\pi k'/k}$$

(1) For $x=1$, $k=k'$, $k=k'=\frac{1}{\sqrt{2}}$ etc.

(2) For $x=\sqrt{3}$, $2kk'=\sin\frac{\pi}{6}$; vide: Greenhill; *Elliptic Functions*, p. 331.

(3) For $x=\sqrt{7}$, $2kk'=\frac{1}{8}$, Greenhill p. 331.

(4) For $x=\sqrt{15}$, $\sqrt[4]{kk'}=\sin 18$; Greenhill has $\sqrt{kk'}=\sin 18$, an obvious misprint.

(6) $f(x)+f(x^{-1})=\frac{1}{2}(\sin^{-1}k+\sin^{-1}k')=\frac{\pi}{4}$.

Question 530.

(N. Sankara Aiyar, M.A.) :—Solve in integers

$$\frac{x^2+17}{y+9}=\text{a perfect square.}$$

Solution by A. Narasinga Rao.

Let $$\frac{x^2+17}{y+9}=z^2 \qquad \dots \qquad \dots \qquad \dots (1)$$

Put $y+9=y$, so that $x^2+17=y_1\,z^2$.

Give x any integral value n, and write (n^2+17) in terms of its prime factors. Say

$$(n^2+17)=p_1^{\,2\alpha_1+\beta_1}\,p_2^{\,2\alpha_2+\beta_2} \qquad \dots \qquad \dots \qquad \dots (2)$$

where $p_1, p_2 \dots$ are the prime factors of (n^2+17) and the β's are either 0 or 1.

It is obvious from (2) that we may take for z any one of the terms in the distributed product $(1+p_1+p_1^2+\dots p_1^{\alpha_1})(1+p_2+p_2^2+\dots p_2^{\alpha_2})\dots$ If any of these values of z be substituted in (2), we obtain the corresponding value of y_1 and thence that of y. The aggregate of all these sets of values for all integral values of x constitute the complete solution.

A simple solution is

$$z=1,\ x=n,\ y=n^2+8.$$

The question may also be solved by giving values to y_1, and solving the equation

$$x^2-y_1z^2=-17,$$

by the usual methods.

Question 533.

(Prof. K. J. Sanjana):—Three real numbers a, b, c, are written down at random. Find the chance (i) that $b^2<4\,ac$, (ii) that $b^2>4\,ac$, what is the chance of b^2 being equal to $4\,ac$?

Solution by N. Sankara Aiyar, M. A.

Case (i) has already been solved on p. 67, Vol. VI, of J. I. M. S.

Case (ii). If a and c have opposite signs, then $b^2>4ac$. The chance of this is $\frac{1}{2}$.

If a & c have the same sign, if a is greatest the condition reduces to $b^2>4c$; if c is greatest, to $b^2>4a$; and if b is greatest, to $1>4ac$ where a, b, c are $\angle 1$. The chance of each of these cases is $\frac{1}{3}$. In case a is greatest, c ranges from $\frac{b^2}{4}$ to 0 and b ranges from 1 to 0. Hence the chance of $b^2>4c$ in this case is

$$\int_0^1 db \int_0^{\frac{b^2}{4}} dc = \left[\frac{b^3}{12}\right]_0^1 = \frac{1}{12}.$$

Similarly $\frac{1}{12}$ is the chance when c is greatest.

When b is greatest, if $c<\frac{1}{4}$, $4\,ac<1$. The chance of this is clearly $\frac{1}{4}$. When $c>\frac{1}{4}$, a ranges from $\frac{1}{4c}$ to 0 and c from 1 to $\frac{1}{4}$. The chance of $1>4\,ac$ is

$$\int_{\frac{1}{4}}^1 dc \int_0^{\frac{1}{4c}} da = \left[\tfrac{1}{4}\log c\right]_{\frac{1}{4}}^1 = \tfrac{1}{2}\log 2.$$

Hence the total chance of b^2 being greater than $4ac$ is

$$\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{12}+\frac{1}{12}+\frac{1}{4}+\frac{1}{2}\log 2\right)$$
$$=\frac{1}{2}+\frac{5}{72}+\frac{1}{12}\log 2=\frac{41}{72}+\frac{1}{12}\log 2.$$

Hence the chance of $b^2=4ac$ is zero. This is explained by the fact that for any given value of b^2 and of a there is only one out of an infinity of values which will give $b^2=4ac$, and so the chance is $\frac{1}{\infty}$; the chance of $b^2=4ac$ for any given value of b is therefore $\frac{1}{2}\cdot\frac{1}{\infty}$ since a & c should have the same sign. That is, the chance of $b^2=4ac$ is $\frac{1}{2}\,\frac{1}{\infty}$ or zero *i.e.*, negligible.

Question 545.

(M. T. Naraniengar, M.A.) :—Show that the roots of $8x^3-18x+9=0$ and $8y^3-18y-9=0$ are connected by the relations

$$x=3+2y-2y^2 \text{ and } y=3-x-2x^2.$$

Solution by N. Sankara Aiyar, M.A.

We have
$$8x^3-18x+9=0$$
$$8y^3-18y-9=0.$$

Adding,
$$8(x^3+y^3)-18(x+y)=0.$$
$$\therefore \quad 4(x^2-xy+y^2)-9=0; \text{ since } x+y\neq 0.$$
i.e.
$$4x^2-4xy+4y^2-9=0.$$
i.e.
$$(2x-y)=\pm\sqrt{9-3y^2}\ldots\ldots(1)$$

Now
$$(2y-3)(8y^3-18y-9)$$
$$=16y^4-24y^3-36y^2+36y+27=0$$
$$\therefore \quad 16y^4-24y^3-39y^2+36y+36=(9-3y^2)$$
i.e.
$$(4y^2-3y-6)^2=(9-3y^2).$$
$$\therefore \quad 2x-y=4y^2-3y-6 \text{ or } 6+3y-4y^2 \text{ by (1)}$$
i.e.
$$x=2y^2-y-3 \text{ or } 3+2y-2y^2.$$

Similarly, $(2y-x)=\pm\sqrt{9-3x^2}$

And
$$(2x+3)(8x^3-18x+9)$$
$$=16x^4+24x^3-36x^2-36x+27$$
$$=16x^4+24x^3-39x^2-36x+36-(9-3x^2)$$
$$=(4x^2+3x-6)^2-(9-3x^2).$$
$$\therefore \quad 2y-x=\pm(6-3x-4x^2)$$
i.e.
$$y=3-x-2x^2 \text{ or } 2x^2+2x-3.$$

Question 546.

(S. Ramanujan) :—Shew that

(i) $$\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)+\frac{2\cdot4}{3\cdot5^2}\left(\frac{1}{3}-\frac{1}{4^3}\right)+=\frac{\pi}{12}\log(2+\sqrt{3});$$

(ii) $$1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\ldots=\frac{\pi^2}{8}-\frac{1}{2}\{\log(1+\sqrt{2})\}^2.$$

Solution by M. Bhimasena Rao.

(i) $$\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)-\frac{2.4}{3.5^2}\left(\frac{1}{3}-\frac{1}{5^2}\right)+\cdots$$
$$=\frac{1}{3}\left(1+\frac{2}{3^2}+\frac{2.4}{3.5^2}+\cdots\right)-\frac{1}{2}\left\{\frac{1}{2}+\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^5}+\cdots\right\}$$

$$=\frac{1}{3}\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\cdot\frac{dx}{x}-\frac{1}{2}\int_0^{\frac12}\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}\frac{dx}{x}.$$

Now

$$\int_0^x \frac{\sin^{-1}x}{\sqrt{1-x^2}}.\,\frac{dx}{x}=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}-\int_0^x\log\frac{x}{1-\sqrt{1-x^2}}\,d\sin^{-1}x$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}-\int_0^{\sin^{-1}x}\log\frac{\sin\theta}{1+\cos\theta}\,d\theta,\ \text{put }\sin\theta=x$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}+\int_0^{\sin^{-1}x}\log\cot\frac{\theta}{2}.\,d\theta$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}+\int_0^{\sin^{-1}x}2\left(\cos\theta+\frac{\cos3\theta}{3}+\frac{\cos5\theta}{5}+\dots\right)d\theta$$

$$=\sin^{-1}x\log\frac{x}{1+\sqrt{1-x^2}}+2\left(\sin\theta+\frac{\sin3\theta}{3^2}+\frac{\sin5\theta}{5^2}+\dots\dots\right)$$

where $\sin\theta=x$.

$$\therefore\qquad\int_0^1\frac{\sin^{-1}x}{\sqrt{1-x^2}}\cdot\frac{dx}{x}=2\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\dots\right)$$

and

$$\int_0^{\frac12}\frac{\sin^{-1}1x}{\sqrt{1-x^2}}\cdot\frac{dx}{x}=\frac{\pi}{6}\log\frac{1}{2+\sqrt3}+2\left(\sin\frac{\pi}{6}+\frac{\sin\frac{3\pi}{6}}{3^2}+\frac{\sin\frac{5\pi}{6}}{5^2}+\dots\right)$$

$$=\frac{\pi}{6}\log\frac{1}{2+\sqrt3}+2\left(\frac12+\frac{1}{3^2}+\frac12\cdot\frac{1}{5^2}-\frac12\cdot\frac{1}{7^2}-\frac{1}{9^2}-\frac12\cdot\frac{1}{11^2}\dots\right)$$

$$=\frac{\pi}{6}\log\frac{1}{2+\sqrt3}+2\left\{\frac12\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}\dots\right)+\frac12\left(\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}-\dots\right)+\left(\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}\dots\dots\right)\right\}$$

$$=\frac{\pi}{6}\log\frac{1}{2+\sqrt3}+\frac43\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\dots\right)$$

$$\therefore\ \frac13\int_0^1\frac{\sin^{-1}x\,dx}{x\sqrt{1-x^2}}-\frac12\int_0^{\frac12}\frac{\sin^{-1}x\,dx}{x\sqrt{1-x^2}}$$

$$=\frac23\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\dots\right)-\frac{\pi}{12}\log\frac{1}{2+\sqrt3}-\frac23\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\dots\right)$$

$$=\frac{\pi}{12}\log\left(2+\sqrt3\right)$$

(ii) Consider the series

$$x-\frac{2}{3^2}.x^3+\frac{2\cdot4}{3\cdot5^2}.x^5-\ldots\ldots$$

This is equal to

$$\int_0^\pi \frac{\sinh^{-1}x}{x\sqrt{1+x^2}}.dx$$

$$=\int_0^{\sinh^{-1}x}\frac{ydy}{\sinh y}=\int_0^{\sinh^{-1}x} yd\log\tanh\frac{y}{2}$$

$$=2\int_0^{\tanh\frac{\sinh^{-1}x}{2}}\frac{\tanh^{-1}z}{z}dz=2\int_0^{\tanh\frac{\sinh^{-1}x}{2}}\sum\frac{z^{2n+1}}{(2n+1)^2}$$

$$\equiv 2\ f(z) \text{ where } z=\frac{\sqrt{1+x^2}-1}{x}$$

It may be shown that

$$f(z)+f\left(\frac{1-z}{1+z}\right)=\frac{\pi^2}{8}-\tfrac{1}{2}\log z.\log\frac{1-z}{1+z}\ldots \qquad \ldots (1)$$

Putting $z=\frac{1-z}{1+z}$, we get $z=\sqrt{2}\pm1$,

Taking the value $\sqrt{2}-1$, $x=1$ and by (1)

$$f(\sqrt{2}-1)=\frac{\pi^2}{16}-\tfrac{1}{4}\{\log(\sqrt{2}-1)\}^2$$

$$=\frac{\pi^2}{16}-\tfrac{1}{4}\{\log(1+\sqrt{2}\}^2.$$

$$\therefore\ 1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}=\ldots=2f(\sqrt{2}-1)$$

$$=\frac{\pi^2}{8}-\tfrac{1}{2}\{\log(1+\sqrt{2})\}^2.$$

Mr. N. Sankara Aiyar sends a solution of (ii) *only on similar lines.*

Question 575.

(S. Narayana Aiyar, M.A.):—If B_1, B_3, B_5, B_7, &c., are the Bernoulle's numbers $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}$, &c., show that

$$B_1-B_3+B_5-B_7+B_9-B_{11}+\ldots \text{ to } \infty=\frac{\pi^2-9}{6}$$

Solution (1) *by R. S. Pocock, B.A., B.Sc., F.R.A.S., and*
(2) *by K. B. Madhava.*

(1) We have the usual formula in the calculus of finite differences

viz. $$\sum_{1}^{x-1} u_x = C + \int u_x dx - \frac{1}{2}u_x + \frac{B_1}{2!} \cdot \frac{du_x}{dx} - \frac{B_3}{4!} \cdot \frac{d^3u_x}{d_x^3} + \dots$$

Writing $u_x = \frac{1}{x^2}$ and adding $\frac{1}{x^2}$ to each side,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \frac{1}{x^2} = C - \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{x^2} - \frac{B_1}{x^3} + \frac{B_3}{x^5} - \frac{B_5}{x^7} + \dots$$

Putting $x=\infty$, we find $C = \frac{\pi^2}{6}$; and putting $x=1$, we get

$$B_1 - B_3 + B_8 - B_5 + \dots = \frac{\pi^2 - 9}{6}.$$

(2) This is at once deduced from formula (29) obtained by Mr. S. Ramanujan on p. 224 of J. I. M. S., Dec. 1911 (Vol. 3) which states except for notation about the suffixes of B

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \dots \infty = \frac{1}{n} - \frac{1}{2n^2} + \frac{B_1}{n^3} - \frac{B_3}{n^5} + \frac{B_5}{n^7} \dots \infty$$

With $n=1$ and the known value for $\sum_{1}^{\infty} \frac{1}{n^2}$ *viz.* $\frac{\pi^2}{6}$, we have at once

$$B_1 - B_3 + B_5 \dots = \frac{\pi^2 - 9}{6}.$$

This is also obtained from Ex: 26, p. 489 of Bromwich *viz.*,

$$\frac{1}{1+x} + \frac{1}{2(1+2x)} + \frac{1}{3(1+3x)} + \dots = C + \log\frac{1}{x} + \frac{x}{2} - B_1\frac{x^2}{2} + B_3\frac{x^4}{4} \dots$$

By differentiation; which is exactly identical with Mr. Ramanujan's; except that this result can be established by applying Euler's formula (§131, Bromwich) to the function $f = \frac{1}{t(1+xt)}$.

Question 576.

(S. NARAYANA AIYAR, M. A.) :—Shew that

$$e^x = \frac{1 + x\frac{a}{b} + \frac{x^2}{1 \cdot 2}\,\frac{a(a+1)}{b(b+1)} + \dots\dots \text{to } \infty}{1 + x\frac{a-b}{b} + \frac{x^2}{1 \cdot 2}\,\frac{(a-b)(a-b-1)}{b(b+1)} + \dots\dots \text{to } \infty}$$

Solution (1) *by T. P. Trivedi, M.A., LL.B. and*
(2) *by M. Bhimasena Rao.*

(1) $1 + x.\frac{a}{b} + \frac{x^2}{1 \cdot 2}\,\frac{a(a+1)}{b(b+1)} + \dots\dots \text{to } \infty$ is the coefficient of z^{a-1} in

$$\underline{|b-1}\left\{ (z+x)^{a-b} + \frac{(z+x)^{a-b+1}}{\underline{|1}} + \dots\dots \frac{(z+x)^{a-1}}{\underline{|b-1}}\right.$$

$$+\frac{(z+x)^a}{\underline{|b}}+\frac{(z+z)^{a+1}}{\underline{|b+1}}+\ldots\ldots\ldots\ldots\ldots\text{to }\infty\Big\}$$

i.e. in $$\underline{|b-1}(z+x)^{a-b}\left\{1+\frac{z+x}{\underline{|1}}+\frac{(z+x)^2}{\underline{|2}}+\text{ad in} f.\right\}$$

i.e. in $$\underline{|b-1}(z+x)^{a-b}e^{x+z}$$

$$=\underline{|b-1}e^x\times\text{coeff of }z^{a-1}\text{ in}(z+x)^{a-b}e^z=\underline{|b-1}e^x\times\text{coeff of }z^{a-1}\text{ in}$$

$$\{z^{a-b}+{}^{(a-b)}c_1z^{a-b-1}x+{}^{(a-b)}c_2x^{a-b-2}+\ldots\ldots\}\times$$

$$\left\{1+z+\frac{z^2}{\underline{|2}}+\ldots\ldots\frac{z^{b-1}}{\underline{|b-1}}+\frac{x^b}{\underline{|b}}+\frac{z^{b+1}}{\underline{|b+1}}+\ldots\ldots\ldots\right\}$$

$$=\underline{|b-1}e^x\left\{\frac{1}{\underline{|b-1}}+{}^{(a-b)}c_1\,x.\frac{1}{\underline{|b}}+{}^{(a-b)}c_2\,x^2\frac{1}{\underline{|b-2}}+\ldots\ldots\ldots\right\}$$

$$=e^x\left\{1+x.\frac{a-b}{b}+\frac{x^2}{1\cdot2}\frac{(a-b)(a-b-1)}{b(b+1)}+\ldots\ldots\ldots\right\}$$

whence the result

If $a-b$ be an integer, the series in the *den*^{r} will be finite, but this restriction is not necessary for the question.

Quest. 616 suggests another method for obtaining the result. It may also be obtained by multiplying the denominator by the series for e^x and identifying the coefficients.

(2) If $b>a>o$, we have

$$\frac{a(a+1)(a+2)\ldots\ldots(a+n-1)}{b(b+1)(b+2)\ldots\ldots.(b+n-1)}=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 t^{a+n-1}(1-t)^{b-a-1}dt.$$

$$\therefore\ 1+x\frac{a}{b}+\frac{x^2}{2\,!}\,\frac{a(a+1)}{b(b+1)}+\ldots\ldots\ldots$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1\Big\{t^{a-1}(1-t)^{b-a-1}+t^a(1-t)^{b-a-1}x+$$

$$t^{a+1}(1-t)^{b-a-1}-\frac{x^2}{2\,!}+\ldots\Big\}dt$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 t^{a-1}(1-t)^{b-a-1}e^{xt}dt$$

$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 (1-t)^{a-1}t^{b-a-1}e^{x(1-t)}\,dt$$

$$=e^x.\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 t^{(b-a)-1}(1-t)^{a-1}e^{-xt}\,dt$$

$$=e^x\left\{1-\frac{(b-a)}{b}x+\frac{(b-a)(b-a+1)}{b(b+1)}\,\frac{x^2}{2\,!}-\ldots\ldots\right\},$$

by expanding e^{-xt} and integrating

Hence $$e^x=\frac{1+x\frac{a}{b}+\frac{x^2}{2\,!}\,\frac{a(a+1)}{b(b+1)}+\frac{x^3}{3\,!}\,\frac{a(a+1)(v+2)}{b(b+1)(b+2)}+\ldots\ldots}{1+\frac{a-b}{b}\,x+\frac{(a-b)(a-b-1)}{b(b+1)}\,\frac{x^2}{2\,!}+\ldots\ldots}$$

Question 579.

(S. Narayanan, B. A., L. T.) :—With the three vertices of a $\triangle ABC$ as centres circles are described to touch the opposite sides. Shew that the common chords of these circles with the circumcircle form a triangle which is in perspective with the original triangle.

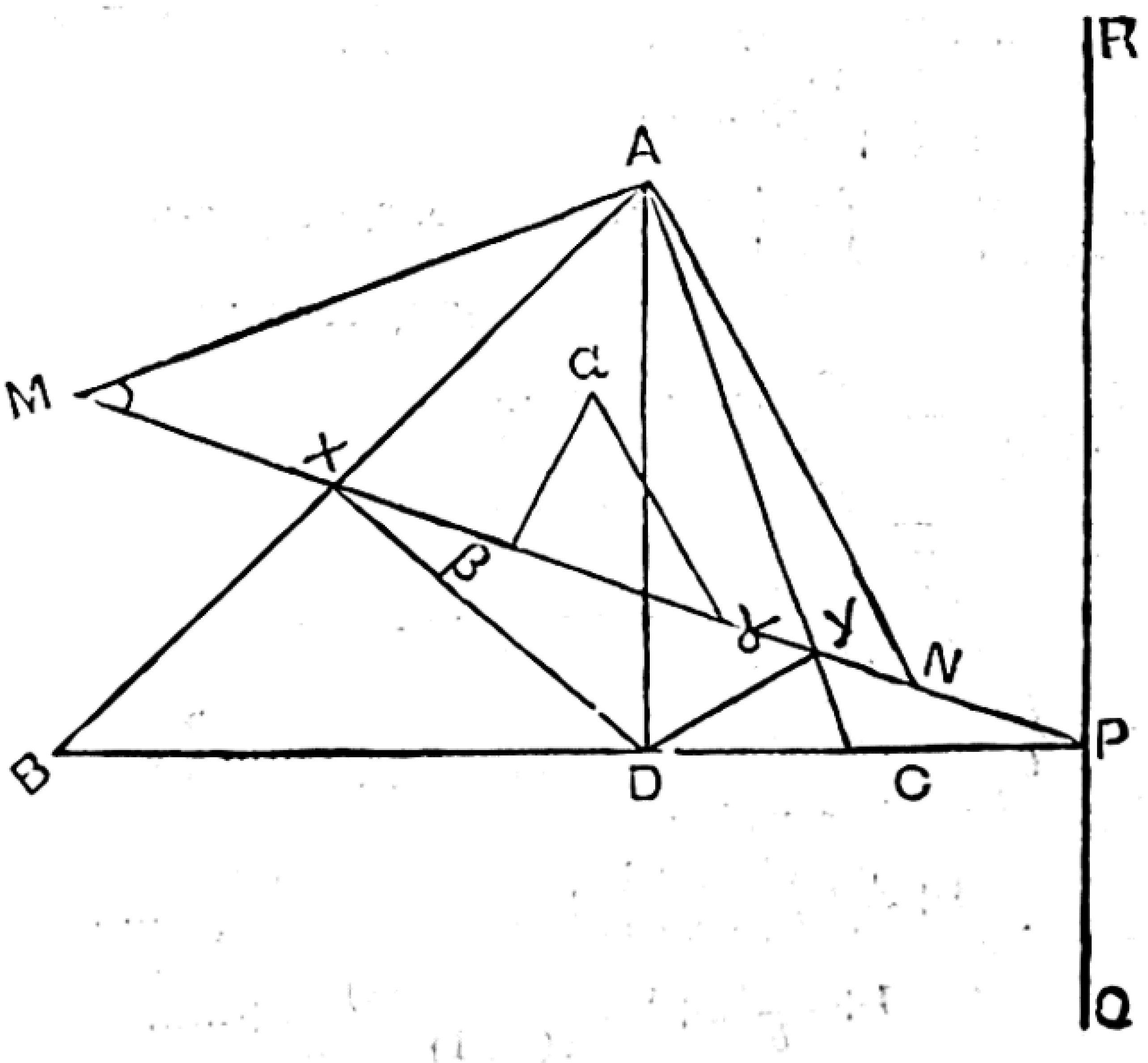
Solution (1) *by V. V. S. Narayan* ; (2) *by D. Krishnamurti and N. Sankara Aiyar,*

(1) Let $\alpha\beta\gamma$ be the Δ formed by the common chords, and let the corresponding sides of $\alpha\beta\gamma$ and ABC intersect at P, Q, R. We shall prove that P, Q, R are collinear. In the figure AD is $\perp$ to BC and $AM=AN=AD$, by hypothesis.

$\therefore$ $A\hat{M}N=A\hat{N}M=A\hat{B}M$.
$\therefore$ $AX.\,AB=AM^2=AD^2$.
$\therefore$ DX is $\perp$ to AB.
Similarly DY is $\perp$ to AC.
Thus $BX/XA=DB^2/DA^2$, from similar triangles.
and $AY/YC=DA^2/CD^2$.
$\therefore$ $BX/XA.\,AY/YC=DB^2/CD^2$.
But $BX/XA.\,AY/YC.CP/PB=1$, by Menelau's Theorem.
$\therefore$ $CP/PB=CD^2/DB^2$.

If DEF is the pedal Δ, we find similarly,

$$BR/RA=BF^2/FA^2,$$
$$AQ/QC=AE^2/EC^2.$$



Compounding,

$$CP/PB.\ BR/RA.\ AQ/QC = CD^2/DB^2.\ BF^2/FA^2.\ AE^2/EC^2 = 1,$$

since AD, BE, CF are concurrent.

Hence P, Q, R, are collinear. That is, the Δ^s ABC, $\alpha\beta\gamma$ are coaxial. But coaxial Δ^s are also copolar. Hence Δ^sABC, $\alpha\beta\gamma$ are copolar.

(2) The equation in areal co-ordinates to a circle of radius r can be written in the form

$$f^2x+g^2y+h^2z=r^2+S,$$

where fgh are the distances of the centre from A,B,C, and $S=0$ represents the equation to the circum-circle: [*Vide* Askwith's *Analytical Geometry*, p. 313, Q. 22.]

The equation to the circle touching BC and having A for centre is thus

$$c^2y+b^2z=p_1^2+S,$$

p_1 being the $\perp^r$ from A on BC.

The equation to the common chord of this and the circumcircle is

$$c^2y+b^2z=p_1^2(x+y+z) \qquad \dots \qquad \dots (1)$$

The co-ordinates of the point L of intersection of (1) with BC, or $x=0$ are proportional to

$$\begin{aligned} x:y:z &= 0 : -(b^2-p_1^2) : c^2-p_1^2 \\ &= 0 : -b^2(1-c^2\lambda^2) : c^2(1-b^2\lambda^2) \\ &= 0 : -C'b^2 : B'c^2, \end{aligned}$$

where $\sin A/a = \dots = \lambda$, and $A'=(1-a^2\lambda^2)$, etc.

Similarly the other points M and N can be got.

Since the determinant formed by the co-ordinates of L, M, N, *viz.*,

$$\begin{vmatrix} 0 & -C'b^2 & B'c^2 \\ Ca^2 & 0 & A'c^2 \\ -Ba^2 & Ab^2 & 0 \end{vmatrix}$$

is identically zero, L, M, N are collinear and the result follows.

Question 582.

(J. C. Swaminarayan, M. A.):—Prove the identities

$$\tan^{-1}x-\tan^{-1}\frac{x}{3}+\tan^{-1}\frac{x}{5}-\tan^{-1}\frac{x}{7}+\dots\dots=\tfrac{1}{2}\,\mathrm{gd}\frac{\pi}{2}x$$

and

$$\tanh^{-1}x-\tanh^{-1}\frac{x}{3}+\tanh^{-1}\frac{x}{5}\dots\dots=\tfrac{1}{2}\,\mathrm{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

Solution (1) *by K. B. Madhava and N. Sankara Aiyar;*
(2) *by T. P. Trivedi, M. A., LL. B.*

(1) Now

$$\frac{1}{1^2-x^2}-\frac{3}{3^2-x^2}+\frac{5}{5^2-x^2}-\frac{7}{7^2-x^2}+\dots\dots$$

$$=\frac{\pi}{4}\left\{\frac{\pi}{\frac{\pi^2}{4}(1^2-x^2)}-\frac{3\pi}{\frac{\pi^2}{4}(3^2-x^2)}+\frac{5\pi}{\frac{\pi^2}{4}(5^2-x^2)}+\dots\dots\right\}.$$

$$=\frac{\pi}{4}.\ \sec\frac{\pi}{2}x$$ (see Bromwich : *Infinite Series*, p 190, Ex. 19.)

Integrating each side we have

$$\tanh^{-1}x-\tanh^{-1}\frac{x}{3}+\tanh^{-1}\frac{x}{5}\dots\dots=\tfrac{1}{2}\ \mathrm{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

Again, writing ix for x in the above

$$\frac{1}{1^2+x^2}-\frac{3}{3^2+x^2}+\frac{5}{5^2+x^2}-\frac{7}{7^2+x^2}\quad+\dots\dots$$

$$=\frac{\pi}{4}\left\{\frac{\pi}{\frac{\pi^2}{4}(1^2+x^2)}-\frac{3\pi}{\frac{\pi^2}{4}(3^2+x^2)}+\frac{5\pi}{\frac{\pi^2}{4}(5^2+x^2)}\dots\dots\right\}$$

$$=\frac{\pi}{4}\operatorname{sech}\left(\frac{\pi}{2}x\right).$$

Integrating each side, we have

$$\tan^{-1}x-\tan^{-1}\frac{x}{3}+\tan^{-1}\frac{x}{5}-\tan^{-1}\frac{x}{7}+\dots=\tfrac{1}{2}\ \mathrm{gd}\left(\frac{\pi}{2}x.\right)$$

(2) We have

$$\sin(x+iy)=(x+iy)\prod_{1}^{\infty}\left\{1-\frac{(x+iy)^2}{n^2\pi^2}\right\};$$

taking logarithms,

$$\log(\sin x\cosh y+i\ \sinh y\cos x)=\log(x+iy)+\log\left\{1-\frac{x+iy}{\pi}\right\}$$

$$+\log\left\{1+\frac{x+iy}{\pi}\right\}+\log\left\{1-\frac{x+iy}{2\pi}\right\}$$

$$+\log\left\{1+\frac{x+iy}{2\pi}\right\}+\dots$$

Put $x=\frac{\pi}{4}$ and $y=\frac{\pi x}{4}$, and equate the imaginary parts on both sides, we get

$$\tan^{-1}(\tanh\frac{\pi}{4}x)=\tan^{-1}\frac{\frac{\pi x}{4}}{\frac{\pi}{4}}+\tan^{-1}\frac{-\frac{\pi x}{4}}{\frac{3\pi}{4}}+\tan^{-1}\frac{\frac{\pi x}{4}}{\frac{5\pi}{4}}+\text{ etc.}$$

$$=\tan^{-1}x-\tan^{-1}\frac{x}{3}+\tan^{-}\frac{x}{5}-\mathrm{txn}^{-1}\frac{x}{7}+\text{etc.}$$

But $$2\ \tan^{-1}\left(\tanh\frac{\pi}{4}x\right)=\tan^{-1}\frac{2\ \tanh\frac{\pi}{4}x}{1-\tanh^2\frac{\pi}{4}x}$$

$$=\tan^{-1}\left(\sinh\frac{\pi}{2}x\right)=\mathrm{gd}\left(\frac{\pi}{2}x\right)\ ;$$

whence the first result follows.

Again

$$\frac{1}{2}\text{gd}^{-1}\left(\frac{\pi}{2}x\right)=\frac{1}{2}\log\ \tan\left(\frac{\pi}{4}+\frac{\pi}{4}x\right)$$

$$=\frac{1}{2}\left\{\log\ \sin\left(\frac{\pi}{4}+\frac{\pi}{4}x\right)-\log\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\ \right)\right\}$$

$$=\frac{1}{2}\left\{\ \log\ \left(\frac{\pi}{4}+\frac{\pi}{4}x\right)+\log\left[1-\frac{\frac{\pi^2}{16}(1+x)^2}{\pi^2}\right]\right.$$

$$+\log\left[1-\frac{\frac{\pi^2}{16}(1+x)^2}{2^2\pi^2}\right]+\log\left[1-\frac{\frac{\pi^2}{16}(1+x)^2}{3^2\pi^2}\right]$$

$$+\text{etc}-\log\left[1-\frac{\frac{4\pi^2}{16}(1+x)^2}{\pi^2}\right]-\log\left[1-\frac{\frac{4\pi^2}{16}(1+x)^2}{3^2\pi^2}\right]$$

$$\left.-\log\left[1-\frac{\frac{4\pi^2}{16}(1+x)^2}{5^2\pi^2}\right]-\text{etc.}\right\}$$

$$=\frac{1}{2}\left\{\ \log\ \frac{\pi}{4}+\log\ \ (1+x)+\log\ (3-x)+\log\ (5+x)-\log\ 4^2\right.$$

$$+\log\ (7-x)+\log\ (9+x)-\log 8^2+\text{etc}-\log\ (3+x)$$

$$-\log\ (1+x)+\log\ 2^2-\log\ (5+x)-\log\ (7+x)$$

$$\left.+\log 6^2-\text{etc}\ \right\}$$

$$=\frac{1}{2}\left\{\ \log\ \frac{\pi}{4}+\log\frac{2^2.\ 6^2.\ 10^2\ldots\ldots}{4^2.\ 8^2.\ 12^2\ldots\ldots}+\log\frac{1+x}{1-x}\right.$$

$$\left.-\log\frac{3+x}{3-x}+\log\frac{5+x}{5-x}-\log\frac{7+x}{7-x}+\text{etc.}\right\}.$$

And

$$\log\frac{2^2.\ 6^2.\ 10^2\ldots\ldots}{4^2.\ 8^2.\ 12^2\ldots\ldots}=-\log\frac{2^2.\ 4^2.\ 6^2\ldots\ldots}{1^2.\ 3^2.\ 5^2\ldots\ldots}=-\log\ \frac{\pi}{4},$$

by employing Wallis' formula;

also

$$\frac{1}{2}\log\frac{1+x}{1-x}=\tanh^{-1}x\ ;$$

$$\frac{1}{2}\log\frac{3+x}{3-x}=\frac{1}{2}\log\frac{1+\frac{x}{3}}{1-\frac{x}{3}}=\tanh^{-1}\frac{x}{3}\ ;\ \text{etc.}$$

$$\tanh^{-1}x-\tanh^{-1}\frac{x}{3}+\tanh^{-1}\frac{x}{5}-\text{etc}=\frac{1}{2}\text{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

The second part may also be obtained readily by changing x into xi in the first.

Question 583.

(J. C. SWAMINARAYAN, M. A.) :—Show that

$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1+x)}{x}dx=\frac{\pi^2}{15}-\frac{1}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

and

$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1-x)}{x}dx=\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2-\frac{\pi^2}{10}.$$

(Suggested by Question 572 of MR. BHIMASENA RAO.)

Solution by K. B. Madhava and N. Sankara Aiyar, M. A.

Let $$\phi(x)=x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+\dots$$

Then $$\phi(x)=\int^x-\frac{\log(1-x)}{x}dx.$$

and $$\phi(-x)=\int^x-\frac{\log(1+x)}{x}dx.$$

so that we have

(i) $$\phi(x)+\phi(-x)=\int^x-\frac{\log(1-x^2)}{x}dx=\frac{1}{2}\phi(x^2).$$

Also we have

(ii) $$\phi(x)+\phi(1-x)=\int\left\{\frac{\log x}{1-x}-\frac{\log(1-x)}{x}\right\}dx$$
$$=\frac{\pi^2}{6}-\log x\log(1-x)$$

the constant being found by putting $x=1$.

Finally we have

(iii) $$\phi(-x)+\phi\left(\frac{x}{1+x}\right)=\int-\frac{\log(1+x)}{x}+\frac{\log(1+x)}{x(1+x)}dx.$$
$$=-\frac{1}{2}\left\{\log 1+x\right\}^2.$$

Now, if we put $x=\frac{1}{2}(\sqrt{5}-1)$, $x^2+x-1=0$;

$$1-x=\frac{x}{1+x};\ (1+x)=\frac{1}{x},\ \text{and } 1-x=x^2\ \textit{etc.}$$

Hence

$$\phi(x)=\frac{\pi^2}{10}-(\log x)^2.$$

$$\phi(x^2)=\frac{\pi^2}{15}-(\log x)^2.$$

$$\phi(-x)=\frac{1}{2}(\log x)^2-\frac{\pi^2}{15}.$$

which give the required results.

QUESTIONS FOR SOLUTION.

646. (R. Srinivasan, M.A.) :—If $xy+yz+zx=1$, show that

$$\sum\left\{(1+x^2)^{\frac{1}{2}}-x\right\}\left\{(1+y^2)^{\frac{1}{2}}-y\right\}=1.$$

647. (K. Appukuttan Erady, M.A.) :—The curves $\alpha=$constant, $\beta=$constant, where α, β are conjugate functions of (x, y) pass through a point. Prove that the curvatures of the two curves at the point are

$$-\frac{\partial h}{\partial \alpha} \text{ and} -\frac{\partial h}{\partial \beta},$$

where

$$h=\left\{\frac{\partial(\alpha,\beta)}{\partial(x,y)}\right\}^{\frac{1}{2}}$$

648. (K. Appukuttan Erady, M.A.) :—Tangents TP, TQ are drawn to the ellipse $x^2/a^2+y^2/b^2=1$, and the parallelogram TPSQ is completed. Prove that the area of the locus of S, as T completes a circuit along $x^2/\lambda^2+y^2/\mu^2=1$, is

$$\pi\lambda\mu+2\pi\, ab(a^2/\lambda^2+b^2/\mu^2-2).$$

649. (S. Malhari Rao, B.A.) :—A gentleman has one third of an acre of vacant land round his house. He wishes to divide it into five different plots to grow five different kinds of flowers. What must be the areas of these plots so that they may be in A. P. and each of them may be contained an exact number of times in an acre? Shew that there cannot be more than one set of values for the areas.

650. (N. P. Pandya) :—Circumscribe an ellipse about a given triangle so that the centre of the ellipse may coincide with the in-centre of the triangle.

651. (S. Malhari Rao, B. A.) :—The Nasik or pan-diagonal Magic Square (*a*) has been constructed by placing the series of natural numbers from 1 upto 25 along the combined paths of a Knight and a Rook in chess. By adopting another method, and guided by the altered positions of 1, 7, 13, 19 and 25 as shown in square (*b*), fill up the vacant cells in it so that it may be magic and pan-diagonal.

(*a*)

12	25	8	16	4
18	1	14	22	10
24	7	20	3	11
5	13	21	9	17
6	19	2	15	23

(*b*)

13				
	19			
		25		
			1	
				7

652. (S. P. Singaravelu Mudaliar) :—From a point (eccentric angle ϕ) of an ellipse of semi-axes a, b, three normals are drawn to the ellipse, shew that the square on the radius of the circle passing through the feet of the normals is

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

653. (Selected) :—Find a cube the sum of whose divisors is a perfect square.

654. (R. Vythynathaswamy) :—A plane moves in any manner on a fixed plane and returns to its original position after n revolutions. If A, B, C, D be the areas described by four points a, b, c, d in the moving plane, prove the identical relation

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & n\pi \\ 1 & 0 & ab^2 & ac^2 & ad^2 & A \\ 1 & ba^2 & 0 & bc^2 & bd^2 & B \\ 1 & ca^2 & cb^2 & 0 & cd^2 & C \\ 1 & da^2 & db^2 & dc^2 & 0 & D \\ n\pi & A & B & C & D & 0 \end{vmatrix}=0.$$

Shew that if we put $\frac{1}{2}\pi$ for $n\pi$ in this, we get the identical relation between the mutual distances of the four points and the areas of their pedals with respect to any closed curve.

Also prove from this relation that if $A=B=C=D$, then the four points lie on a circle.

655. (A. Narasinga Rao) :—Solve for f from the relation

$$f^{(n)}(x)=\alpha f(x+\beta)$$

α and β being any constants.

A curve and its n^{th} evolute are similar. Find the intrinsic equation of the curve.

656. (A. Narasinga Rao) :—A square is divided into n^2 smaller squares by lines $\|$ to the sides. P,Q are any two points within the square. What is the average number of lines which the finite line PQ meets? What is the chance that it should meet just k lines?

657. (T. P. Trivedi, M.A., LL.B.) :—Prove that the value of the determinant of the $(n+2)^{th}$ order

$$\begin{vmatrix} 0, & 1, & 1, & 1, & \cdots & \cdots & 1 \\ 1, & 0, & a_1+a_2, & a_1+a_3 & \cdots & \cdots & a_1+a_{n+1} \\ 1, & a_1+a_2 & 0, & a_2+a_3 & \cdots & \cdots & a_2+a_{n+1} \\ 1, & a_1+a_{n+1}, & a_2+a_{n+1}, & a_3+a_{n+1} & \cdots & \cdots & 0 \end{vmatrix}$$

is equal to

$$-(-2)^n a_1 a_2 a_3 \ldots a_{n+1} \left\{ \frac{1}{a_1} + \frac{1}{a_2} + \ldots \frac{1}{a_{n+1}} \right\}.$$

Also find the value of the determinant of the $(n+1)^{th}$ order, obtained from the above by deleting the first column and the first row.

658. (K. J. SANJANA, M. A.) :—Prove the identity

$$\begin{aligned} &2\{ax+by+(a+b)(x+y)\}^3 \\ &-6\{ax+by+(a+b)(x+y)\}(a^2+ab+b^2)(x^2+xy+y^2) \\ &-27abxy(a+b)(x+y)=(a-b)(x-y)(2a+b)(a+2b)(2x+y)(x+2y)\,; \end{aligned}$$

and deduce the theorem of Cayley that

$$ax+by+cz,\ bx+cy+az,\ cx+ay+bz$$

are the three roots of the equation

$$\begin{aligned} &4t^3-3(a^2+b^2+c^2)(x^2+y^2+z^2)t-54\,abcxyz \\ &\qquad -2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)=0, \end{aligned}$$

when $a+b+c$ and $x+y+z$ both vanish.

[In the *Messenger of Mathematics* for November 1914, Prof. Steggall has given a direct proof of the theorem which, it appears, was proved by Cayley by the use of determinants. The question suggests a very simple indirect proof.]

659. (M. BHIMASENA RAO) :—Shew that

$$(1)\ \left(1+e^{-\pi}\right)\left(1+e^{-2\pi}\right)\left(1+e^{-3\pi}\right)\ldots\ldots=e^{\pi/24}\left(\frac{1}{2}\right)^{\frac{1}{8}}$$

$$(2)\ \left(1+e^{-\pi\sqrt{2}}\right)\left(1+e^{-2\pi\sqrt{2}}\right)\left(1+e^{-3\pi\sqrt{2}}\right)\ldots= e^{\pi\sqrt{2}/24}\left(\frac{1}{2}\right)^{\frac{1}{4}}.$$

660. (M. BHIMASENA RAO) :—Shew that the locus of the double point of orthologic triangles described on AB and CD is the radical axis of the circles having AD and BC as diameters.

661. (S. RAMANUJAN) :—Solve in integers

$$x^3+y^3+z^3=u^6,$$

and deduce the following :

$$\begin{aligned} &6^3-5^3-3^3=2^6\,; \\ &8^3+6^3+1^3=3^6\,; \\ &12^3-10^3-1^3=3^6\,; \\ &46^3-37^3-3^3=6^6\,; \\ &174^3+133^3-45^3=14^6\,; \\ &1188^3-509^3-3^3=34^6. \end{aligned}$$

662. (S. RAMANUJAN) :—Let AB be a diameter and BC be a chord of a circle ABC. Bisect the minor arc BC at M ; and draw a chord BN equal to half of the chord BC. Join AM. Describe two circles with A and B as centres and AM and BN as radii cutting each other at S and S′ and cutting the given circle again at the points M′ and N′ respectively. Join AN and BM intersecting at R and also join AN′ and BM′ intersecting at R′. Through B draw a tangent to the given circle meeting AM and AM′ produced at Q and Q′ respectively. Produce AN and M′B to meet at P and also produce AN′ and MB to meet at P′. Show that the eight points PQRSS′R′Q′P′ are cyclic and that the circle passing through these points is orthogonal to the given circle ABC.

663. (S. Krishnaswami Aiyangar, B. A.):—Establish the following relation with respect to Bernoulli's numbers :—

$$\frac{n}{2}=c_2.\,\mathrm{B}_1-c_4.\,\mathrm{B}_3.\,4+c_6.\,\mathrm{B}_5.\,4^2-(-)^{n-1}c_{2n}.\,\mathrm{B}_{2n-1}.4^{n-1}$$

where $c_r = 2_{n+1}Cr$

664. (S. Krishnaswami Aiyangar, B. A.) :—Prove that

$$\sum_1^{\infty}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2 \cdot \frac{1}{n}=\frac{16}{\pi}-4.$$

665. (R. N. Apte, M.A., F.R.A.S.) :—Find the value of

$$\iint xyz\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}^{\frac{1}{2}} dx\,dy$$

where $z\equiv(1-x^2/a^2-y^2/b^2)^{\frac{1}{2}}$ and the integration is over the positive quadrant of $x^2/a^2+y^2/b^2=1$.

666. (S. Ramanujan) :—Solve in positive rational numbers

$$x^y=y^x.$$

For example : $x=4$, $y=2$; $x=3\frac{3}{8}$, $y=2\frac{1}{4}$.

667. (S. Narayana Aiyar, M. A.) :—Establish the truth of the following theorems :

1. If $\mathrm{F}(y)=\phi(y)=\frac{x}{1!}\phi(y+1)+\frac{x^2}{2!}\phi(y+2)-\frac{x^2}{3!}\phi(y+3)+\ldots$

then $\phi(y)=\mathrm{F}(y)+\frac{x}{1!}\mathrm{F}(y+1)+\frac{x^2}{2!}\mathrm{F}(y+2)+\frac{x^3}{3!}\mathrm{F}(y+3)+\ldots$

2. If $\mathrm{F}(y)=\phi(y)-\frac{n}{1}\phi(y+1)+\frac{n(n+1)}{1\cdot2}\phi(y+2)-\ldots$

then $\phi(y)=\mathrm{F}(y)+\frac{n}{1}.\mathrm{F}(y+1)+\frac{n(n-1)}{1\cdot2}\mathrm{F}(y+2)+\ldots$

List of Periodicals Received.

(From 16th March to 15th May 1915.)

1. American Journal, of Mathematics, Vol. 37, No. 1, January 1915.
2. Astrophysical Journal, Vol. 41, No. 1, January 1915.
3. Bulletin of the American Mathematical Society, Vol. 21, No. 5, February 1915.
4. Bulletin des Sciences Mathematiques, Vol. 39, January and February 1915.
5. Educational Times, March and April 1915, (6 Copies.)
6. L'Intermediaire des Mathematiciens, Vol. 21, Nos. 11 & 12, November and December 1915.
7. Mathematical Gazette, Vol. 8, No. 115, January 1915, (4 copies)
8. Messenger of Mathematics, Vol. 44, No. 9, January 1915.
9. Monthly Notices of the Royal Astronomical Society Vol. 75, No. 3, January 1915.
10. Philosophical Magazine, Vol. 29, No. 171, March 1915.
11. Popular Astronomy, Vol. 23, Nos. 2 & 3, February and March 1915, (3 copies).
12. Proceedings of the London Mathematical Society, Vol. 14, No. 3, March 1915.
13. Proceedings of the Royal Society of London, Vol. 91, No 627, March 1915.
14. Transactions of the Cambridge Philosophical Society, Vol. 22, Nos. 6 & 7, February 1915.
15. American Mathematical Monthly, Vol. 22, No. 1, January 1915.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts of the Society for the half year ended 30th June 1915.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Opening Balance	508	10	2	Books and Periodicals...	332	10	2
Subscriptions from Members	1,355	11	0	Printing Journal	276	0	6
Subscriptions for Journal	149	0	0	Library	120	0	0
Other receipts	22	11	0	Working Charges	102	3	3
				Closing Balance*	1,205	2	3
Total ...	2,036	0	2	Total ...	2,036	0	2

	Rs.	A.	P.
* With the Treasurer	946	2	0
" Secretary	28	6	6
" Assistant Librarian ...	126	0	0
" " Secretary ...	104	9	9
Total ...	1,205	2	3

MADRAS,
30*th July* 1915.

S. NARAYANA AIYAR,
Hon. Treasurer.

Forces along non-intersecting Generators of a Hyperboloid.

By R. VYTHYNATHASWAMY, B.A.

[In this paper some properties of systems of forces along generators of the same system of a hyperboloid are deduced by combining the theory of the *nul point* and the *nul plane* with that of the *polar point* and the *polar plane*. A brief account of such systems of forces is to be found in Routh's *Analytical Statics*, Vol. I, Chapter VII, to which the reader is referred.]

1. *The identical relation between the mutual moments of seven lines in space.*

If $(l_r, m_r, n_r, \lambda_r, \mu_r, \nu_r)(r=1, 2, \ldots 7)$ be the co-ordinates of the lines and (rs) represent the moment of $(l_r \ldots)$ and $(l_s \ldots)$, it is readily seen by the multiplication of the arrays

$$\|l_r\, m_r\, n_r\, \lambda_r\, \mu_r\, \nu_r\|, \; \|\lambda_r\, \mu_r\, \nu_r\, l_r\, m_r\, n_r\| \qquad (r=1, 2 \ldots 7)$$

that the required relation is

$$\begin{vmatrix} 0 & (12) & (13) & (14) & (15) & (16) & (17) \\ (21) & 0 & (23) & (24) & (25) & (26) & (27) \\ (31) & (32) & 0 & (34) & (35) & (36) & (37) \\ (41) & (42) & (43) & 0 & (45) & (46) & (47) \\ (51) & (52) & (53) & (54) & 0 & (56) & (57) \\ (61) & (62) & (63) & (64) & (65) & 0 & (67) \\ (71) & (72) & (73) & (74) & (75) & (76) & 0 \end{vmatrix} = 0.$$

2. *The necessary and sufficient condition that the skew lines a, b, c, d lie on a hyperboloid is*

$$\Delta \equiv \begin{vmatrix} 0 & (ab) & (ac) & (ad) \\ (ba) & 0 & (bc) & (bd) \\ (ca) & (cb) & 0 & (cd) \\ (da) & (db) & (dc) & 0 \end{vmatrix} = 0$$

or

$$p^2+q^2+r^2-2qr-2rp-2pq=0,$$

where $p=(bc)(ad)$, $q=(ca)(bd)$, $r=(ab)(cd)$.

First :—The condition is necessary. For, if a, b, c, d lie on a hyperboloid, then there are three lines e, f, g which meet them all. The identical relation of the previous article becomes

$$\Delta \times \begin{vmatrix} 0 & (ef) & (eg) \\ (fe) & 0 & (fg) \\ (ge) & (gf) & 0 \end{vmatrix} = 0.$$

Since e, f, g are skew, the second determinant cannot vanish and hence

$$\Delta \equiv 0.$$

To prove the second part ; Let e, f be the determinate lines which meet a, b, c, d and g an indeterminate line meeting a, b, c. (Routh, § 320). Since Δ is given to vanish, the fundamental identical relation assumes the form

$$(dg)^2\phi_1 + (dg)\phi_2 = 0,$$

where ϕ_1, ϕ_3 do not depend on g.

Hence (dg) is either zero or constant for all positions of g.

In the latter case also (dg) is zero, since e, f are possible positions of g.

Thus g meets d and hence a, b, c, d lie on a hyperboloid.

3. *If four non-intersecting forces are in equilibrium their lines of action lie on a hyperboloid.* (Routh, § 316.)

Let P, Q, R, S be the forces, a, b, c, d their lines of action.

Taking moments about a, b, c, d and eliminating P, Q, R, S from the resulting equations, we see that the determinant Δ of the last article vanishes. Thus a, b, c, d lie on a hyperboloid.

Cayley's Theorem :—The forces P, Q, R, S are proportional to $A^{\frac{1}{2}}, B^{\frac{1}{2}}, C^{\frac{1}{2}}, D^{\frac{1}{2}}$ where $A = (bc)(cd)(db)$, etc. (Routh, § 316, Ex. 6.)

Let I_{rs} represent the minor of the constituent of the r^{th} row and s^{th} column in the determinant Δ. Then

$$\frac{P}{I_{11}} = \frac{Q}{I_{12}} = \frac{R}{I_{13}} = \frac{S}{I_{14}}.$$

But by Salmon's *Higher Algebra* $I_{11} \cdot I_{22} = I^2_{12}$.

$$\therefore \qquad \frac{P^2}{I_{11}} = \frac{Q^2}{I_{22}} = \frac{R^2}{I_{33}} = \frac{S^2}{I_{44}}.$$

Since $I_{11} = A, \ldots$ the result follows.

Cor.—Resolving along the four lines, we get

$$A^{\frac{1}{2}} + B^{\frac{1}{2}} \cos (\hat{ab}) + C^{\frac{1}{2}} \cos (ac) + D^{\frac{1}{2}} \cos (\hat{ad}) \equiv 0,$$

and three similar relations.

It is easily seen, that the forces are also proportional to the lines of the solid angles between every three of the four lines.

4. *Nul Lines and Planes* :

Consider a system of forces along generators of the λ-system of a hyperboloid.

(i) Every μ—generator is a *nul line* of the system since it meets all the forces.

(ii) The nul plane of any point A on the hyperboloid passes through the μ—generator through A and is therefore the tangent plane to the hyperboloid at some point B of this generator.

(iii) The system can be reduced to two conjugate forces along generators of the λ—system. (Routh, § 303.)

For the conjugate of a λ—generator must meet all the nul lines which meet that generator. That is, it must meet all the μ—generators and is consequently a λ—generator.

Def. Two such generators shall be called *conjugate generators* of the system.

5. *If the nul plane of A is the tangent plane at B, then the nul plane of B is the tangent plane at A.*

Let a, a' be the λ—and μ—generators through A, and let b be the generator conjugate to a. Then the nul plane of A passes through b and is the plane (Ab) or ($a'b$) and is thus the tangent plane at the intersection of a' and b. Hence B is the intersection of a' and b.

Similarly, the nul plane of B is the plane (Ba) or (aa') and is therefore the tangent plane at A.

Def :—Two points on the hyperboloid may be said to be *conjugate*, when the nul plane of each is the tangent plane at the other. Thus it appears from the above that any two conjugate generators cut any μ-generator in conjugate points.

6. *If P,P′ be the ends of a diameter of the hyperboloid and Q,Q′ their conjugates, then QQ′ will be parallel to the principal force of the system.*

For the nul planes of Q and Q′ must be parallel, which means that the tangent planes at the conjugates P, P′ should be parallel. Hence the result.

7. *Self-conjugate points and self-conjugate generators* :—If QQ′ be itself a diameter, then PP′ and QQ′ will coincide. In other words, the principal force at the centre meets the hyperboloid in *self-conjugate points*. The λ—generator of either of these self-conjugate points is a *self-conjugate generator*; every point of a self-conjugate generator is a self-conjugate point, since conjugate generators must pass through

conjugate points. The *self-conjugate generators* are nul lines of the system, since they lie in the nul planes of the self-conjugate points.

There is some difficulty in the proper conception of a self-conjugate generator. A self-conjugate generator does not mean that the given system of forces reduces to a single force along it, as may appear at first sight. When two conjugate generators coincide what happens really is that the equivalent forces along these become infinite and by pulling in opposite directions along the self-conjugate produce the same effect as a finite force and a couple. (See Routh, § 306.)

8. *The nul plane at the centre is the diametral plane of the principal force.* (Routh, § 317, Ex. 9.)

For the nul plane at the centre is parallel to the nul plane of any other point on the principal force at the centre and is thus parallel to the nul plane at the self-conjugate point in which the principal force meets the hyperboloid. Now, this latter is the tangent plane at the self-conjugate point, and therefore the parallel central plane has the property stated.

9. The result of § 6 may be generalized as follows :—*If PP' be a chord of the hyperboloid passing through a fixed point L, and (Q, Q') the conjugates of (P, P'), then the chord QQ' passes through another fixed point, M, which is the pole of the nul plane of L.*

For, the nul planes of points on a line have a common line of intersection (Routh § 303.) Therefore the nul planes of P, P′ intersect in the nul plane of L. But the nul planes of P, P′ are the tangent planes at Q, Q′. Hence the tangent planes at Q, Q′ intersect in the nul plane of L. By reciprocation, the chord QQ′ passes through the pole of the nul plane of L, which is a fixed point.

Cor. Let R be the principal force at the centre and R_1, R_2 two conjugate diameters in the diametral plane of R; then if PP′ be a chord parallel to R_1, QQ′ will be parallel to R_2. For, in this case L may be regarded as the point at infinity on R_1, whose nul plane is the plane (RR_1); and therefore QQ′ always passes through the pole of the plane (RR_1), *viz*: the point at infinity on R_2, that is QQ′ is always parallel to R_2.

10. *Cross ratio* :—The cross ratio of four points on a line is equal to the cross ratio of their nul planes with respect to any set of forces. For, the nul planes of the four points pass through the same conjugate line and are cut by the given line at these points. The following involution properties are noteworthy :

(i) *Pairs of conjugate generators are in involution, that is, they cut any μ—generator in a range in involution.*

Let pp', qq', rr', be pairs of conjugate generators and let a μ-generator cut them in PP′, QQ′, RR′, which are pairs of conjugate points. Then

$$\begin{aligned}(PQRP') &= \text{cross ratio of nul planes of } PQRP' \\ &= \text{cross ratio of tangent planes at } P'Q'R'P \\ &= (P'Q'R'P.)\end{aligned}$$

Thus PP′, QQ′, RR′ belong to the same involution.

Hence pp', qq', rr' are in involution.

(ii) *The two self-conjugate generators and any two conjugate generators form a harmonic set; that is, they intercept on any μ-generator a harmonic range.*

For, the self-conjugates are the double lines of the involution to which pairs of conjugate generators belong.

(iii) *Pairs of conjugate points on a μ-generator belong to an involution of which the self-conjugate points are the double points.*

N.B.—The truth of the above theorems, may also be inferred by conceiving the hyperboloid as a one-dimensional space the elements of which are λ-generators, in as much as the correspondence of conjugacy is reciprocal.

11. *The central axis:—The central axis of the system is the shortest distance between the two μ-generators which are perpendicular to the principal force.*

Let the central axis intersect the hyperboloid in P, Q and let p, q be the μ-generators through P, Q.

Since the nul planes of P, Q are perpendicular to the central axis, the lines p, q which lie in these planes are also perpendicular to the central axis, and therefore to the principal force.

Hence the central axis is the shortest distance between p and q.

Some further Generalizations:

12. *If L, M be any two points (not necessarily on the hyperboloid) such that the nul plane of L is the polar plane of M, then the nul plane of M is the polar plane of L.*

For, it has been proved in § 9 that, if PP′ be any chord through L, and (Q, Q′) the conjugates of (P, P′) then the chord QQ′ passes through M the pole of the nul plane of L. Since the relations of L and M are reciprocal, the result stated follows at once.

Def. Two points L, M such that the nul plane of each is the polar plane of the other may be said to be ***conjugate*** with respect to our system of forces. Also two lines such as PP′, QQ′ may be called ***corresponding lines*** of the system.

[*Note.* This definition of conjugate points does not conflict with the definition in § 5 above. For, if L is on the hyperboloid, M will be its conjugate as previously defined.]

13. *Two corresponding lines are such that the 'conjugate line' of either is the 'polar line' of the other.*[*]

Let PP′, QQ′ be the corresponding lines and l, m, their conjugate lines. Then, the nul plane of any point L on PP′ passes through l and is the polar plane of a point M on QQ′ (§ 12); and the polar line of QQ′ is the common section of the polar planes of different points on it. Hence l is the polar line of QQ′.

Cor. 1. Conjugate λ—generators are corresponding lines of the system.

Cor. 2. The locus of points conjugate to points on a given line is the corresponding line.

Cor. 3. To find the line corresponding to a given line: let the given line cut the hyperboloid in P, P′; join their conjugate points Q, Q′; QQ′ is the corresponding line required.

Cor. 4. If PP′ and QQ′ coincide, we have *a self-corresponding line.* This may happen in either of two ways:

(i) P may coincide with Q and P′ with Q′; in which case the self-corresponding line is a line joining any two *self-conjugate* points.

This includes the case of *self-conjugate generators* also.

(ii) P may coincide with Q′ and P′ with Q, in which case the self-corresponding line is the line joining *any point* on the hyperboloid to *its conjugate* and is a μ—generator.

Hence a self-corresponding line *generally* meets the self-conjugate generators.

14. *The line joining any point and its conjugate (not necessarily on the hyperboloid) is a self-corresponding line.*

Let L, M be a pair of conjugate points. Then the polar line of LM is the intersection of the polar planes of L and M, which are the nul planes of M and L respectively. Thus the polar line of LM is identical with the conjugate line. Hence LM is a self-corresponding line.

15. *A pair of corresponding lines and the self-conjugate generators form generators of the same system of another hyperboloid.*

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[*] The expression 'polar line' is here used to denote the common line of intersection of the *polar planes* of points on a given line, to distinguish it from the line of action of the conjugate force which is here called 'the conjugate line'.

For, the join of a pair of conjugate points, one on each of the corresponding lines is a self-corresponding line and therefore meets the self-conjugate generators. Hence, the four lines in question form generators of the same system of a certain hyperboloid and the joining line is a generator of the opposite system.

16. Consider a hyperboloid formed as above, with the self-conjugates and a given pair of corresponding lines as its λ—generators. It is evident that the corresponding line of any λ—generator of this hyperboloid will be another λ—generator.

To prove that pairs of such corresponding λ—generators are in involution.

Let pp', qq', rr' be three such pairs of corresponding λ—generators. Let them cut a self-corresponding line in PP′, QQ′, RR′. These form pairs of conjugate points and we have

$$\begin{aligned}(PQRP') &= \text{cross ratio of nul planes of } PQRP',\\ &= \text{cross ratio of polar planes of } P'Q'R'P,\\ &= (P'Q'R'P.)\end{aligned}$$

Hence (PP′, QQ′, RR′) and therefore (pp', qq', rr') are in involution.

The self-conjugates are the double lines of this involution, and we may therefore say that a pair of corresponding lines and the self-conjugate generators form a *harmonic* system as in § 10.

Similarly, pairs of conjugate points on a self-corresponding line belong to an involution the double points of which are the self-conjugate points on the line.

17. *If (pp', qq') are two pairs of skew lines, there is a unique pair of lines (rr') which separates both of them harmonically.*

For, since the hyperboloids ($pp'rr'$), ($qq'rr'$) have two generators of one system common they have also two generators of the opposite system common. These are the two lines which meet (pp', qq').

Hence the following construction for (rr'). (Routh, § 320.) Let the two lines meeting (pp', qq') cut them in (P_1P_1', Q_1Q_1') (P_2P_2', Q_2Q_2') respectively.

Then rr' are the lines which divide these sets of points harmonically.

Cor.—Two systems of forces acting along λ—generators have in general only one common conjugate line-pair.

For, the only pair of corresponding lines common to both systems is the line-pair separating harmonically the self-conjugates of both systems. There is only one such pair and it lies on the hyperboloid, as shown above. Hence the only pair of common corresponding lines is the pair of conjugate generators separating both the self-conjugates harmonically.

18. We shall conclude with a few properties of the general system of forces:

(i) *Any two pairs of conjugate lines with respect to a general system of forces are generators of the same system of a certain hyperboloid.*

Let (PP′), (QQ′) be the pairs of conjugate forces; then the forces P, P′, —Q, —Q′ acting along the same lines are in equilibrium. Hence, these lines of action lie on a hyperboloid. (Routh, § 316.)

(ii) *Any two systems of forces have one conjugate line-pair in common.* (Routh, § 311, Ex.)

Through any point there passes a common nul line to both systems, *viz.*, the line of intersection of the nul planes of that point with respect to the two systems. Describe a hyperboloid with three such common nul lines for μ—generators. The given systems of forces can now be reduced to forces along λ—generators of this hyperboloid. Hence the theorem reduces to what has been proved in § 17, *Cor.* above.

(iii) *The common nul lines of three systems of forces generate a hyperboloid.*

Let A, B, C be the three systems; (pp'), (qq') the common conjugate pairs of (A and B) and (B and C) respectively. Then (pp', qq') lie on a hyperboloid by (i). The generators of the opposite system are the common nul lines of the three systems.

(iv) *Four systems of forces have in general two nul lines in common.*

For, if D be the fourth system and (rr') the common conjugate pair of C and D, the hyperboloids $(pp'qq')$, $(qq'rr')$, have two generators of one system common. Therefore they have two generators of the other system in common. These two meet (pp', qq', rr') and are the common nul lines required.

* * *

On the Number of Divisors of a Number.

By S. Ramanujan.

1. If δ be a divisor of N, then there is a conjugate divisor δ' such that $\delta\delta' = N$. Thus we see that

(1) the number of divisors from 1 to $\sqrt{N}$ is equal to the number of divisors from $\sqrt{N}$ to N.

From this it evidently follows that

$$(2) \qquad d\,(N) < 2\sqrt{N},$$

where d (N) denotes the number of divisors of N (including unity and the number itself). This is only a trivial result as all the numbers from 1 to $\sqrt{N}$ cannot be divisors of N. So let us try to find the best possible superior limit for $d(N)$ by using purely elementary reasoning.

2. First let us consider the case in which all the prime divisors of N are known. Let

$$N = p_1^{a_1} . p_2^{a_2} . p_3^{a_3} . \ldots \qquad \ldots p_n^{a_n},$$

where $p_1, p_2, p_3 \ldots p_n$ are a given set of n primes. Then it is easy to see that

$$(3) \qquad d\,(N) = (1+a_1)(1+a_2)(1+a_3)\ldots(1+a_n).$$

But

$$(4) \quad \frac{1}{n}\Big\{(1+a_1)\log p_1 + (1+a_2)\log p_2 + \ldots + (1+a_n)\log p_n\Big\}$$
$$> \Big\{(1+a_1)(1+a_2)\ldots(1+a_n)\log p_1 \log p_2 \ldots \log p_n\Big\}^{1/n}$$

since the arithmetic mean of unequal positive numbers is always greater than their geometric mean. Hence

$$\frac{1}{n}\Big\{\log p_1 + \log p_2 + \ldots + \log p_n + \log N\Big\}$$
$$> \Big\{\log p_1 \log p_2 \ldots \log p_n . d(N)\Big\}^{1/n}$$

In other words

$$(5) \qquad d(N) < \frac{\Big[\frac{1}{n}\Big\{\log(p_1 p_2 p_3 \ldots\ldots p_n N)\Big\}\Big]^n}{\log p_1 \log p_2 \log p_3 \ldots\ldots \log p_n},$$

for all values of N whose prime divisors are $p_1, p_2, p_3 \ldots\ldots p_n$.

3. Next let us consider the case in which only the number of prime divisors of N is known. Let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots \qquad \ldots \qquad \ldots p_n^{a_n},$$

where n is a given number; and let

$$N' = 2^{a_1} . 3^{a_2} . 5^{a_3} \ldots \qquad \ldots \qquad \ldots p^{a_n},$$

where p is the natural nth prime. Then it is evident that

$$N' \le N; \tag{6}$$

and
$$d(N') = d(N). \tag{7}$$

But

$$d(N') < \frac{\left\{\frac{1}{n}\log(2.3.5\ldots\ldots p.N')\right\}^{n}}{\log 2 \log 3 \log 5 \ldots\ldots \log p} \tag{8}$$

by virtue of (5). It follows from (6) to (8) that, if p be the natural nth prime, then

$$d(N) < \frac{\left\{\frac{1}{n}\log(2.3.5\ldots\ldots p.N)\right\}^{n}}{\log 2 \log 3 \log 5 \ldots \log p}, \tag{9}$$

for all values of N having n prime divisors.

4. Finally, let us consider the case in which nothing is known about N. Any integer N can be written in the form

$$2^{a_2}.\,3^{a_3}.\,5^{a_5}\ldots\ldots\ldots\ldots,$$

where $a_\lambda \ge 0$. Now let

$$x^h = 2, \tag{10}$$

where h is any positive number. Then we have

$$\frac{d(N)}{N^h} = \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdot \frac{1+a_5}{5^{ha_5}} \ldots\ldots\ldots \tag{11}$$

But from (10) we see that, if q be any prime greater than x, then

$$\frac{1+a_q}{q^{ha_q}} \le \frac{1+a_q}{x^{ha_q}} = \frac{1+a_q}{2^{a_q}} \le 1. \tag{12}$$

It follows from (11) and (12) that, if p be the largest prime not exceeding x, then

$$\begin{aligned} \frac{d(N)}{N^h} &\le \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{3^{ha_3}} \cdot \frac{1+a_5}{5^{ha_5}} \ldots\ldots\ldots \frac{1+a_p}{p^{ha_p}}. \\ &\le \frac{1+a_2}{2^{ha_2}} \cdot \frac{1+a_3}{2^{ha_3}} \cdot \frac{1+a_5}{2^{ha_5}} \ldots\ldots\ldots \frac{1+a_p}{2^{ha_p}}. \end{aligned} \tag{13}$$

But it is to easy to show that the maximum value of $(1+a)\,2^{-ha}$ for the variable a is $\frac{2^h}{(h\, e \log 2)}$. Hence,

$$\frac{d(N)}{N^h} \le \left(\frac{2^h}{h\, e \log 2}\right)^{\omega(x)} \tag{14}$$

where $\omega(x)$ denotes the number of primes not exceeding x. But from (10) we have

$$h = \frac{\log 2}{\log x}.$$

Substituting this in **(14)**, we obtain

$$(15) \qquad d(\mathrm{N}) \leqq \mathrm{N}^{\frac{\log 2}{\log x}} \cdot \left\{ \frac{2^{\frac{\log 2}{\log x}} \log x}{e\,(\log 2)^2} \right\}^{\omega(x)}.$$

But it is easy to verify that, if $x \geqq 6{\cdot}05$, then

$$2^{h} < e(\log 2)^2.$$

From this and (15) it follows that, if $x \geqq 6{\cdot}05$, then

$$(16) \qquad d(\mathrm{N}) < 2^{(\log \mathrm{N})/(\log x)} (\log x)^{\omega(x)}$$

for all values of N, $\omega(x)$ being the number of primes not exceeding x.

5. The symbol "O" is used in the following sense:

$$\phi(x) = \mathrm{O}\,\{\psi(x)\}$$

means that there is a positive constant K such that

$$\left| \frac{\phi(x)}{\psi(x)} \right| \leqq \mathrm{K}$$

for all sufficiently large values of x. [See Hardy, *Orders of Infinity*, pp. 5 *et seq.*] For example:

$$5x = \mathrm{O}(x)\,;\ \tfrac{1}{2}x = \mathrm{O}(x)\,;\ x \sin x = \mathrm{O}(x)\,;\ \sqrt{x} = \mathrm{O}(x)\,;\ \log x = \mathrm{O}(x)\,;$$

but

$$x^2 \neq \mathrm{O}(x)\,;\ x\log x \neq \mathrm{O}(x).$$

Hence it is obvious that

$$(17) \qquad \omega(x) = \mathrm{O}(x).$$

Now, let us suppose that

$$x = \frac{\log \mathrm{N}}{(\log \log \mathrm{N})^2}$$

in (16). Then we have

$$\log x = \log \log \mathrm{N} + \mathrm{O}(\log \log \log \mathrm{N})\,;$$

and so

$$(18) \qquad \frac{\log \mathrm{N}}{\log x} = \frac{\log \mathrm{N}}{\log \log \mathrm{N}} + \mathrm{O}\left\{ \frac{\log \mathrm{N} \log \log \log \mathrm{N}}{(\log \log \mathrm{N})^2} \right\}.$$

Again

$$(19) \qquad \omega(x) \log \log x = \mathrm{O}(x \log \log x)$$

$$= \mathrm{O}\left\{ \frac{\log \mathrm{N} \log \log \log \mathrm{N}}{(\log \log \mathrm{N})^2} \right\}.$$

It follows from (16), (18) and (19), that

$$(20) \qquad d(\mathrm{N}) < 2^{(\log \mathrm{N})/\log \log \mathrm{N} + \mathrm{O}\left\{ \frac{\log \mathrm{N} \log \log \log \mathrm{N}}{(\log \log \mathrm{N})^2} \right\}}$$

for all sufficiently large values of N.

SHORT NOTES.

The Axes of a Conicoid and an Allied Cone.

The following method of determining the axes of the general conicoid may be of interest as it is an extension of the method usually adopted in the case of the general conic

1. Let $\phi\ (x,\ y,\ z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$ be the conicoid referred to rectangular axes through the centre. Then a concentric sphere of radius r will cut the conicoid in sphero-conics lying on the cone

$$\phi-(x^2+y^2+z^2)/r^2=0 \quad \dots \quad \dots \quad \dots (1)$$

When the sphere touches the conicoid, r will be equal to one of the principal semi-axes, and the cone (1) will in this case denote the line of contact of the sphere and the conicoid, that is, one of the principal axes.

The condition for this is obtained by identifying (1) with

$$\alpha\left(\frac{y}{m}-\frac{z}{n}\right)^2+\beta\left(\frac{z}{n}-\frac{x}{l}\right)^2+\gamma\left(\frac{x}{l}-\frac{y}{m}\right)^2=0 \dots \quad \dots (2)$$

where l, m, n are the direction cosines of the line of contact and α, β γ are unknown. Thus, we may write

$$a-1/r^2=(\beta+\gamma)/l^2, \dots\dots$$

$$f=-\alpha/mn, \dots\dots\dots$$

whence we infer

$$\left.\begin{array}{r}(a-\lambda)\ l+h\ m+g\ n=0,\\ h\ l+(b-\lambda)\ m+fn=0,\\ g\ l+f\ m+(c-\lambda)\ n=0,\end{array}\right\} \quad \dots \quad \dots \quad \dots (3)$$

putting $\lambda=1/r^2$.

Eliminating l, m, n we get the usual determinant

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0. \quad \dots \quad \dots (4)$$

2. Equation (4) can also be derived as the condition that (1) should represent two central planes, which must be the circular sections of the conicoid.

The circular sections are therefore written

$$(a-\lambda)\ x+hy+gz=\pm[\ \{\ h^2-(a-\lambda)\ (b-\lambda)\ \}\ y^2$$

$$+2\ \{\ gh-f(a-\lambda)\ \}\ yz+\ \{\ g^2-(a-\lambda)(c-\lambda)\ \}\ z^2]^{\frac{1}{2}}$$

where the expression within the square brackets is a perfect square.

In other words, the circular sections are

$$(a-\lambda)\,x+hy+gz=\pm\,\frac{\{\,[h^2-(a-\lambda)\,(b-\lambda)]\,y+[gh-f\,(a-\lambda)z]\,\}}{\sqrt{h^2-(a-\lambda)\,(b-\lambda)}}$$

3. For all values of r the cone (1) has its principal axes in the same direction as the conicoid ϕ. For, the discriminating cubic of the cone differs from that of the conicoid in having

$$\left(a-\frac{1}{r^2}\right),\ \left(b-\frac{1}{r^2}\right),\ \left(c-\frac{1}{r^2}\right),$$

in place of a, b, c; so that the roots of the former must be

$$\left(\lambda_1-\frac{1}{r^2}\right),\ \left(\lambda_2-\frac{1}{r^2}\right),\ \left(\lambda_3-\frac{1}{r^2}\right)$$

where λ_1, λ_2, λ_3 are the roots of the latter. Hence, the equations for the direction cosines of the axes of the conic will remain the same as (3).

M. T. NARANIENGAR.

The four-point Conic of minimum Eccentricity.

1. Let P, Q, R, S be four given points. Then, we know that 'the axes of the two parabolas through the points are parallel to conjugate diameters of any conic through the same.' [Smith's *Conic Sections*, 1910; p. 292.] Now, in a given ellipse the acute angle between two conjugate diameters is least when they are equi-conjugates, and in different ellipses the angle between the equi-conjugates is greatest when the eccentricity is least. Thus, if the angle between a pair of conjugate diameters is known, the conic will have a minimum eccentricity, when these diameters are the equi-conjugates.

Hence, we have the following: *The conic of minimum eccentricity through four given points has its equi-conjugates parallel to the axes of the two parabolas through the points.*

2. Again 'if the normals to an ellipse at four points are concurrent, the axis of the parabola through the points is parallel to one or other of the equi-conjugates.' (Ex. 3, p. 269). Hence we infer that *this conic of minimum eccentricity is such that the normals at the points P, Q, R, S, are concurrent.*

3. Lastly 'if normals at P, Q, R, S are concurrent and the circle PQR meet the conic in S′, then SS′ is a diameter' (Ex. 2, p. 269). Hence the centre of the conic is the middle point of SS.′

If p, q, r be the mid-points of SP, SQ, SR and p', q', r' of QR, RP, PQ this centre lies on the circle pqr and is obviously the common point of the four circles pqr, $pq'r'$, $p'qr'$, $p'q'r$. Also, the centre of the rectangular hyperbola through P, Q, R, S is the common point of the circles $p'q'r'$, $p'qr$, $pq'r$, pqr'.

Hence, *the centre of the ellipse in question and the centre of the rectangular hyperbola are equidistant from and collinear with the mean centre of the four points.*

N. Durai Rajan.

Notes on Questions 546, 572, 573, 583, 585, 606.

[Most of the results in these questions follow from some formulæ given (very inaccurately) in Carr's *Synopsis.*]

1. Let a be a positive quantity < 1, so that $\dfrac{1}{1+a}$ and $\dfrac{1-a}{1+a}$ are also positive proper fractions. Then $\phi(a) \equiv \sum_1^\infty (a^n \div n^2)$ is convergent; and it is seen that

$$\phi(a) = -\int_0^a \frac{\log(1-x)}{x}dx,\ \phi(-a) = -\int_0^a \frac{\log(1+x)}{x}dx;$$

$$\phi(1) = \frac{\pi^2}{6},\ \phi(-1) = -\frac{\pi^2}{12};\ \phi(a)+\phi(-a) = \tfrac{1}{2}\phi(a^2),$$

and $$\phi(a)-\phi(-a) = \int_0^a \frac{1}{x}\log\frac{1+x}{1-x}dx = 2\int_0^a \frac{\tanh^{-1}x}{x}dx.$$

We shall use $f(a)$ for

$$\int_0^a \frac{\tanh^{-1}x}{x}dx \equiv \sum_0^\infty \left[a^{2n+1} \div (2n+1)^2\right];$$

thus $f(1)$ is $\dfrac{\pi^2}{8}$.

Differentiating $\log x \log(1-x)$, we get at once

$$\Big\{\log x \log(1-x)\Big\}_0^a = -\int_0^a \frac{\log x}{1-x}dx + \int_0^a \frac{\log(1-x)}{x}dx$$

$$= \int_1^{1-a} \frac{\log(1-x)}{x}dx + \int_0^a \frac{\log(1-x)}{x}dx$$

$$= -\phi(1-a) + \phi(1) - \phi(a);$$

so that $$\phi(a)+\phi(1-a) = \frac{\pi^2}{6} - \log a \log(1-a); \qquad \ldots \text{(A)}$$

$\therefore$ Similarly $\left\{\log x \log(1+x) - \tfrac{1}{2}[\log(1+x)]^2\right\}_0^a$

$$= \int_0^a \frac{\log x}{1+x}dx + \int_0^a \frac{\log(1+x)}{x}dx - \int_0^a \frac{\log(1+x)}{1+x}dx$$

$$= -\phi(-a) + \int_0^a \frac{1}{1+x}\log\frac{x}{1+x}dx$$

$$= -\phi(-a) - \int_1^{\frac{1}{1+a}} \frac{\log(1-x)}{x}dx,$$

$$= -\phi(-a) + \phi\left(\frac{1}{1+a}\right) - \frac{\pi^2}{6};$$

so that $\phi\left(\frac{1}{1+a}\right) - \phi(-a) = \frac{\pi^2}{6} + \log a \log(1+a) - \frac{1}{2}[\log(1+a)]^2.$

The left side can also be written

$$\phi(a) - \tfrac{1}{2}\phi(a^2) + \phi\left(\frac{1}{1+a}\right),\ 2f(a) - \phi(a) + \phi\left(\frac{1}{1+a}\right) \quad \ldots\ldots \text{(B)}$$

Again, differentiating $\log x \tanh^{-1}x$, we get

$$\log a \tanh^{-1}a = \int_c^a \frac{\tanh^{-1}x}{x}dx + \int^a \frac{\log x}{1-x^2}dx.$$

Changing x to $\frac{1-x}{1+x}$, we have

$$\int_0^a \frac{\log x\, dx}{1-x^2} = \int_1^{\frac{1-a}{1+a}} \frac{\tanh^{-1}x}{x}dx = f\left(\frac{1-a}{1+a}\right) - f(1);$$

hence $f(a) + f\left(\frac{1-a}{1+a}\right) = \frac{\pi^2}{8} + \frac{1}{2}\log a \log\left(\frac{1+a}{1-a}\right) \quad \ldots$ (C)

When $a = \frac{1}{2} = 1-a$, (A) gives

$$2\phi\left(\tfrac{1}{2}\right) = \frac{\pi^2}{6} - \left(\log \tfrac{1}{2}\right)^2,$$

or
$$\sum_1^{\infty} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \tfrac{1}{2}(\log 2)^2.$$

When $a = \frac{1-a}{1+a}$, so that a is the positive fraction $\sqrt{2}-1$, (C) gives

$$2f(\sqrt{2}-1) = \frac{\pi^2}{8} - \tfrac{1}{2}\left\{\log(\sqrt{2}-1)\right\}^2;$$

this is result (i) in Question 572, which will be found to be identical with result (ii) in Q. 546.

When $a=\frac{1}{1+a}$, so that $a^2+a=1$ and a is the positive fraction $\frac{1}{2}(\sqrt{5}-1)$, (B) gives

$$2\phi\left(\frac{\sqrt{5}-1}{2}\right)-\frac{1}{2}\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{6}-\frac{3}{2}(\log a)^2;$$

and (A) gives

$$\phi\left(\frac{\sqrt{5}-1}{2}\right)+\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{6}-2(\log a)^2;$$

solving these equations, we get

$$\phi\left(\frac{\sqrt{5}-1}{2}\right)\text{ or }\sum_1^\infty\frac{1}{n^2}\left(\frac{\sqrt{5}-1}{2}\right)^n=\frac{\pi^2}{10}-\left(\log\frac{\sqrt{5}-1}{2}\right)^2,$$

and $$\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{15}-\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$$

The first of these is the second result in Question 583.

The last formula in (B) at once gives in this case

$$2f(a)=\frac{\pi^2}{6}-\frac{3}{2}(\log a)^2,$$

i.e., $$f\left(\frac{\sqrt{5}--1}{2}\right)=\frac{\pi^2}{12}-\frac{3}{4}\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$$

This is result (ii) in Question 572, where there is an evident mistake.

Since $$\phi\left(-\frac{\sqrt{5}-1}{2}\right)=\frac{1}{2}\phi\left(\frac{3-\sqrt{5}}{2}\right)-\phi\left(\frac{\sqrt{5}-1}{2}\right),$$

therefore $$-\int_1^{\frac{\sqrt{5}-1}{2}}\frac{\log(1+x)}{x}dx=\frac{1}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2-\frac{\pi^2}{15},$$

which is the first result in Question 583.

Putting $a=\frac{\sqrt{5}-1}{2}$ and remembering that $\frac{1-a}{1+a}=a^3=\sqrt{5}-2$, we see that we have from (C)

$$\begin{aligned}f\left(\frac{\sqrt{5}-1}{2}\right)+f(\sqrt{5}-2)&=\frac{\pi^2}{8}-\frac{1}{2}\log a\log a^3\\&=\frac{\pi^2}{8}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2;\end{aligned}$$

so that $$f(\sqrt{5}-2)=\frac{\pi^2}{24}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2.$$

This may be put into the form

$$\frac{\pi^2}{24}-\frac{3}{4}\left\{\frac{1}{3}\left(\log\frac{\sqrt{5}-1}{2}\right)^3\right\}^2=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(\sqrt{5}-2)\right\}^2;$$

these are the results in Question 572 (iii) and Q. 606, which are thus found to be identical.

3. We have $\displaystyle\int_0^{\frac{\pi}{2}} \tan^{-1}(a \sin\phi)\, d\phi$

$$= a\int_0^{\frac{\pi}{2}} \sin\phi\, d\phi - \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi + \frac{a^5}{5}\int_0^{\frac{\pi}{2}} \sin^6\phi\, d\phi - \ldots$$

$$= a - \frac{2}{3}\cdot\frac{a^3}{3} + \frac{2\cdot4}{3\cdot5}\frac{a^6}{5} - \ldots$$

$$= \int_0^a \left(1 - \frac{2}{3}x^2 + \frac{2\cdot4}{3\cdot5}x^4 - \ldots\right) dx = \int_0^a \frac{\sinh^{-1}x}{x\sqrt{(1+x^2)}}\, dx,$$

by a well-known formula.

Put $x = \dfrac{2z}{1-z^2}$, so that $\sqrt{1+x^2} = \dfrac{1+z^2}{1-z^2}$, and $\dfrac{dx}{dz} = \dfrac{2(1+z^2)}{(1-z^2)^2}$;

also $z = \dfrac{1}{x}(\sqrt{1+x^2}-1)$, so that the limits become $\dfrac{1}{a}(\sqrt{1+a^2}-1)$, and 0.

If now $m = \sqrt{1+a^2} - a$,

$$\frac{1}{m} = \sqrt{1+a^2} + a\ ,\ \ \sqrt{1+a^2} = \frac{1+m^2}{2m},\ \text{and}\ a = \frac{1-m^2}{2m},$$

so that the upper limit becomes $\dfrac{1-m}{1+m}$.

Thus the integral is transformed into

$$\int_0^{\frac{1-m}{1+m}} \frac{1}{z} \sinh^{-1}\left(\frac{2z}{1-z^2}\right) dz,$$

i.e.,

$$\int_0^{\frac{1-m}{1+m}} \frac{2\tanh^{-1}z}{z}\, dz,\ \text{or}\ 2f\left(\frac{1-m}{1+m}\right).$$

Hence, by (C),

$$\int_0^{\frac{\pi}{2}} \tan^{-1}(a\sin\phi)\, d\phi = 2\left(\frac{\pi^2}{8} + \tfrac{1}{2}\log m \log\frac{1+m}{1-m}\right) - 2f(m)$$

$$= \frac{\pi^2}{4} - \log m \log\frac{1-m}{1+m} - 2\sum_0^\infty m^{2n+1} \div (2n+1)^2.$$

This is the result in Question 585, where there is a slight misprint. When $a=1$, $m=\sqrt{2}-1$; and we see that

$$\int_0^{\frac{\pi}{2}} \tan^{-1}(\sin\ \phi)\ d\phi=1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\ldots$$

$$=2f(\sqrt{2}-1)=\frac{\pi^2}{8}-\tfrac{1}{2}\left\{\log(\sqrt{2}-1)\right\}^2,$$

as already deduced above. This is result (ii) in Question 546.

When $a=\frac{1}{2}$, $m=\frac{\sqrt{5}-1}{2}$; and we get

$$\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^8}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\cdots$$

$$=2f(\sqrt{5}-2)=\frac{\pi^2}{12}-\frac{3}{2}\left\{\log\left(\frac{\sqrt{5}-1}{2}\right)\right\}^2.$$

Now the series

$$\frac{1}{3}-\frac{1}{4}-\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)+\frac{2\cdot4}{3\cdot5^2}\left(\frac{1}{3}-\frac{1}{4^8}\right)-\ldots\ldots$$

$$=\frac{1}{3}\left\{1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\ldots\right\}-\frac{1}{2}\left\{\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\ldots\ldots\right\}$$

$$=\frac{2}{3}f(\sqrt{2}-1)-f(\sqrt{5}-2)$$

$$=\frac{3}{4}\left\{\log\left(\frac{\sqrt{5}-1}{2}\right)^2-\frac{1}{6}\left\{(\log(\sqrt{2}-1)\right\}^2,\right.$$

as just proved. This result is analogous to the first result in Question 546.

4. If $S(a)$ is put for the series

$$a+\frac{2}{3}\ \frac{a^3}{3}+\frac{2\cdot4}{3\cdot5}\ \frac{a^5}{5}+\ldots\ldots\ldots ad.\ inf.,$$

which is the expansion of $\int_0^{\frac{\pi}{2}} \tanh^{-1}(a\ \tan\ \phi)d\phi$, we have

$$S(a)=\int_0^a \frac{\sin^{-1}x}{x\sqrt{(1-x^2)}}dx.$$

As in the last paragraph, if $x=\frac{2z}{1+z^2}$, this reduces to $\int_0^b \frac{2\ \tan^{-1}z}{z}\ d,$ where b is the value of z corresponding to the value a of x, *i.e.*, the positive proper fraction $\{1-\sqrt{(1-a^2)}\}/a$.

If $\int_0^b \frac{\tan^{-1}z}{z}dz$ is denoted by $g(b)$, we have

$$g(1)=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots;$$

have put

also $$g(b)=\log b\,\tan^{-1}b-\int_0^b \frac{\log x}{1+x^2}dx$$

$$=\log b\,\tan^{-1}b-\tfrac{1}{2}\int_0^{2\tan^{-1}b} \log(\tan\tfrac{1}{2}\theta)d\theta,\ \{x\equiv\tan\tfrac{1}{2}\theta\}$$

$$=\log b\,\tan^{-1}b+\int_0^{2\tan^{-1}b}\left(\cos\theta+\frac{\cos 3\theta}{3}+\frac{\cos 5\beta}{5}+\dots\right)d\theta$$

$$=\log b\,\tan^{-1}b+\frac{\sin\beta}{1^2}+\frac{\sin 3\beta}{3^2}+\frac{\sin 5\beta}{5^2}+\dots\dots,$$

where $\beta=2\tan^{-1}b=\sin^{-1}a$.

Thus we have $$S(a)=2\,g(b),$$

or $$a+\frac{2}{3^2}a^3+\frac{2\cdot4}{3\cdot5^2}a^5+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}a^7+\dots=\beta\log b+2\sum_0^\infty\frac{\sin(2n+1)\beta}{(2n+1)^2},$$

a result analogous to that in Question 585.

When $\beta=\frac{\pi}{6}$, $a=\frac{1}{2}$, $b=(2-\sqrt{3})$; and we get

$$S(\tfrac{1}{2})=2g\,(2-\sqrt{3})$$

$$=\frac{\pi}{6}\log(2-\sqrt{3})+2\left\{\frac{1}{2}\cdot\frac{1}{1^2}+1.\frac{1}{3^2}+\frac{1}{2}\cdot\frac{1}{5^2}-\frac{1}{2}\cdot\frac{1}{7^2}-1.\frac{1}{9^2}-\frac{1}{2}\cdot\frac{1}{11^2}\dots\right\}$$

$$=\frac{\pi}{6}\log(2-\sqrt{3})+\frac{4}{3}g(1),\text{ on reduction.}$$

Hence $$\frac{2}{3}S(1)-S(\tfrac{1}{2})=-\frac{\pi}{6}\log(2-\sqrt{3})$$

or $$\tfrac{1}{3}S(1)-\tfrac{1}{2}S(\tfrac{1}{2})=\frac{\pi}{12}\log(2+\sqrt{3}),$$

which gives Q. 546 (i) directly.

Putting $\beta=\frac{\pi}{4},\frac{\pi}{3}$ successively and simplifying, we shall get the following identities :

$$1+\frac{2}{3^2}(\tfrac{1}{2})+\frac{2\cdot4}{3\cdot5^2}(\tfrac{1}{2})^2+\dots=-\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$
$$+2\left\{\frac{1}{1^2}+\frac{1}{3^2}-\frac{1}{5^2}-\frac{1}{7^2}+\dots\right\},$$

$$1+\frac{2}{3^2}(\tfrac{3}{4})+\frac{2\cdot4}{3\cdot5^2}(\tfrac{3}{4})^2+\dots=-\frac{\pi}{3\sqrt{3}}\log 3$$
$$+2\left\{\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{11^2}+\dots\right\}.$$

K. J. SANJANA,

Geometrical Proof of a Trigonometrical Identity.

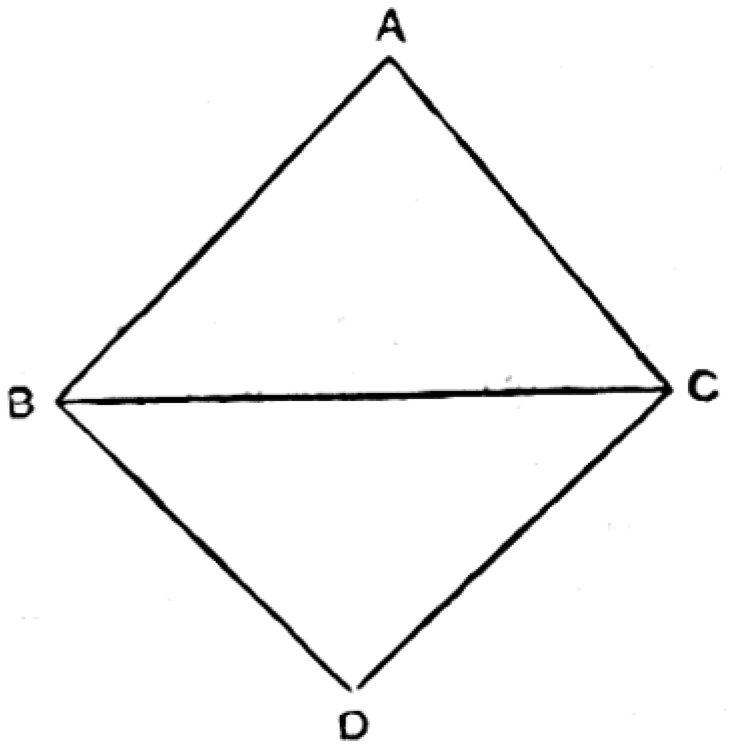
In the *Mathematical Notes* for January 1915 published by the Edinburgh Mathematical Society, Dr. R. J. T. Bell draws attention to a simple geometrical proof of the identity

$$1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C=0,\text{*}$$

where A, B, C are the angles of a triangle.

The following proof appears to be even simpler than the one suggested by Dr. Bell.

Let ABC be any triangle. Draw BD, CD at right angles to **AB**, AC. Then in the triangle BCD



(i) $BD : CD : BC = \sin BCD : \sin CBD : \sin D$
$= \cos C : \cos B : \sin A.$

(ii) $BC^2 = BD^2 + CD^2 - 2\,BD.CD.\cos D$
$= BD^2 + CD^2 + 2BD \cdot CD \cdot \cos A.$

Hence from (i) and (ii)

$$\sin^2 A = \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C,$$

which gives the result.

M. T. Naranienoar.

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* See also *A New Trigonometry* by Lock and Child, p. 344.

SOLUTIONS.

Question 386.

(S. **Ramanujan**):—Shew that

$$\int_0^\infty \frac{dx}{(1+x^2)\,(1+n^2x^2)\,(1+n^4x^2)\ldots} = \frac{\pi}{2(1+n+n^3+n^6+n^{10}+\ldots)}$$

where the indices in the denominator are the sums of the natural numbers.

Solution by N. Durai Rajan.

Let $$I=\int_0^\infty \frac{dx}{(1+x^2)(1+n^2x^2)(1+n^4x^2)(1+n^6x^2)}\cdots$$

and write $$\frac{1}{(1+x^2)\ldots\text{to }\infty}\equiv\frac{a_0}{1+x^2}+\frac{a_1}{1+n^2x^2}+\cdots\frac{a_r}{1+n^{2r}x^2}+\cdots$$

By putting $x^2=-n^{2r}$, we get

$$1/a_r=\left(1-\frac{1}{n^{2r}}\right)\left(1-\frac{n^2}{n^{2r}}\right)\ldots\ldots\left(1-\frac{n^{2r-2}}{n^{2r}}\right)\left(1-\frac{n^{2r+2}}{n^{2r}}\right)\ldots\ldots$$

$$=\left(1-\frac{1}{n^2}\right)\ldots\ldots\left(1-\frac{1}{n^{2r}}\right)(1-n^2)(1-n^4)(1-n^6)\ldots\text{to }\infty$$

$$=\left(1-\frac{1}{n^2}\right)\ldots\ldots\left(1-\frac{1}{n^{2r}}\right)\phi(n)\text{ say}$$

$$\therefore\ I=\sum\int_0^\infty\frac{a_r\,dx}{1+n^{2r}x^2}=\sum\int_0^\infty\frac{a_r}{n^r}\cdot\frac{dy}{1+y^2}.\quad[n^rx=y]$$

$$=\frac{\pi}{2}\sum\frac{a_r}{n^r}.$$

But $$\frac{a_r}{n^r}=\frac{1}{n^r}\cdot\frac{1}{\phi(n)}\cdot\frac{1}{\left(1-\frac{1}{n^2}\right)\ldots\left(1-\frac{1}{n^{2r}}\right)}=\frac{1}{n^r}\cdot\frac{1}{\phi(n)}\,\frac{n^2\cdot n^4\ldots n^{2r}}{(1-n^2)\ldots(1-n^{2r})}(-)^r$$

$$=\frac{1}{\phi(n)}\,\frac{(-1)^r n^{r(r+1)}}{n^r\cdot(1-n^2)\ldots(1-n^{2r})}=\frac{1}{\phi(n)}\cdot\frac{n^{r^2}}{(1-n^2)\ldots(1-n^{2r})}(-)^r$$

$$\therefore\quad\sum\frac{a^r}{n^r}=\frac{1}{\phi(n)}\left[1-\frac{n}{1-n^2}+\frac{n^4}{(1-n^2)(1-n^4)}\right.$$

$$\left.-\frac{n^9}{(1-n^2)(1-n^4)(1-n^6)}+\ldots\ldots\right]$$

$$=\frac{(1-n)(1-n^3)(1-n^5)\ldots\ldots\text{to }\infty}{(1-n^2)(1-n^4)(1-n^6)\ldots\ldots\text{to }\infty}:$$

(Chrystal, Part II, p. 344, Ex. 7.)

$$\therefore\qquad I=\frac{\pi}{2}\,\frac{(1-n)(1-n^3)(1-n^5)\ldots\ldots\text{to }\infty}{(1-n^2)(1-n^4)(1-n^6)\ldots\ldots\text{to }\infty}.$$

But by a theorem of Gauss (See Ex. 10 p: 345, Chrystal)

$$1+\sum x^{\frac{1}{2}n(n+1)}=\frac{(1-n^2)(1-n^4)(1-n^6)}{(1-n)(1-n^3)(1-n^5)}\ldots\infty$$

$$\therefore \qquad I=\frac{\pi}{2(1+n+n^3+n^6+\ldots)}$$

Question 557.

(Professor SANJANA) :—At a point P of a parabola the normal chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that

$$PQ \cdot AN=2\,SP \cdot PG.$$

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$$SP_1 \cdot SP_2=AS.SP,\quad PP_1 \cdot PP_2=PQ \cdot AN$$

where S is the focus.

Shew also that the circle P_1P_2S passes through the circumcentre of PP_1P_2.

Note by the Proposer.

With regard to Mr. V. V. S. Narayan's solution of this Question in the Journal for April last, I have to point out that the first two properties were discovered long ago; the second was published in the *Educational Times* for June 1903, and has been solved in Vol. VI of the *E. T. Reprints* (New Series). The last part was discovered about four years ago.

Question 561.

(P. A. SUBRAMANIYA AIYAR, B.A., L.T.) :—Solve completely

$$(1+3x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-4y=0$$

Solution by A. Narasingu Rao.

Put $y=(2x^2+1)\int u\,dx$. The equation transforms into

$$\frac{du}{dx}+\frac{22x^3+7x}{(2x^2+1)(3x^2+1)}u=0.$$

i.e.,
$$\frac{du}{u}+\left(\frac{8}{2x^2+1}-\frac{1}{3x^2+1}\right)x\,dx=0.$$

$$\therefore \qquad u=\frac{A(3x^2+1)^{\frac{4}{3}}}{(2x^2+1)^2}.$$

$$\therefore \qquad y=A\int\frac{(3x^2+1)^{\frac{4}{3}}}{(2x^2+1)^2}dx+B(2x^2+1),$$

where A and B are arbitrary constants.

Question 585.

(K. V. Ananthanarayan Sastri, B. A.) :—Prove that

$$\int_{0}^{\frac{\pi}{2}} \tan^{-1}(a \sin \phi)\, d\phi = \frac{\pi^2}{4} - \log m \log \frac{1-m}{1+m}$$

$$-2\sum_{0}^{\infty} \frac{m^{2r+1}}{(2r+1)^2},$$

where $m=(1+a^2)^{\frac{1}{2}}-a$.

Solution by T. P. Trivedi M.A., LL.B., and K. B. Madhava.

We have

$$2\sum \frac{m^{2r+1}}{(2m+1)^2} = \int_{0}^{m} \frac{1}{x}\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right)dx$$

$$= \int_{0}^{m} \frac{\tanh^{-1}x}{x}\,dx = 2\Big[\tanh^{-1}x \,\log x\Big]_{0}^{m} - 2\int_{0}^{m} \frac{\log x\, dx}{1-x^2}$$

$$= \log m \log \frac{1+m}{1-m} - 2\int_{1}^{m} \frac{\log x\, dx}{1-x^2} - 2\int_{0}^{1} \frac{\log x\, dx}{1-x^2}$$

$$= \log m \log \frac{1+m}{1-m} - 2\int_{0}^{n} \frac{\log \frac{1-\zeta}{1+\zeta}\, d\zeta}{1-\frac{(1-\zeta)^2}{(1+\zeta)^2}} \times -\frac{2}{(1+\zeta)^2} + \frac{\pi^2}{4}$$

$\left(\text{where in the second integral } n=\frac{1-m}{1+m},\ x=\frac{1-\zeta}{1+\zeta}\right)$,

$$= \log m \log \frac{1+m}{1-m} - 2\int_{0}^{n} \frac{\tanh^{-1}\zeta}{\zeta}\, d\xi + \frac{\pi^2}{4}$$

$$= \log m \log \frac{1+m}{1-m} - \int_{0}^{n} \frac{2 \tanh^{-1}\zeta}{\zeta}\, d\zeta + \frac{\pi^2}{4}$$

$$= \log m \log \frac{1+m}{1-m} - \int_{0}^{a} \frac{2\tanh^{-1}\left(\frac{\sqrt{1+y^2}-1}{y}\right)}{y\sqrt{1+y^2}}\, dy + \frac{\pi^2}{4}$$

$$\text{if } \zeta = \frac{\sqrt{1+y^2}-1}{y}$$

$$= \log m \log \frac{1+m}{1-m} - \int_{0}^{a} \frac{\tanh^{-1}y}{y\sqrt{1+y^2}}\, dy + \frac{\pi^2}{4}$$

$$=\log m \log \frac{1+m}{1-m}+a-\frac{a^3}{3}\cdot\frac{2}{3}+\frac{a^5}{5}\cdot\frac{2\cdot4}{3\cdot5}-\frac{a^7}{7}\cdot\frac{2\cdot4\cdot6}{3\cdot5\cdot7}+\dots+\frac{\pi^2}{4}$$

$$=\log m \log \frac{1+m}{1-m}+\int_0^{\frac{\pi}{2}}\left\{a \sin\phi-\frac{a^3}{3}\sin^3\phi+\dots\right\}d\phi+\frac{\pi^2}{4}$$

$$=\log m \log \frac{1+m}{1-m}+\int_0^{\frac{\pi}{2}}\tan^{-1}(a\sin\phi)\,d\phi+\frac{\pi^2}{4}.$$

Hence the result.

Question 587.

(T. P. TRIVEDI, M. A., L.L.B.) :—If the centre of pressure of a triangle immersed in a liquid coincides with the symmedian point prove that the depths of the angular points are in the ratios

$$3a^2-b^2-c^2 : 3b^2-c^2-a^2 : 3c^2-a^2-b^2.$$

Solution by D. Krishnamurti, M. K. Veeraraghavan and K. B. Madhava.

This question is only a particular case of the following :—

A triangular lamina is totally immersed in a homogeneous liquid the depths of the angular points being p, q, r : shew that if the centre of pressure of the triangle coincide with the mean centre of its angular points for multiples l, m, n, then $p : q : r = 3l-m-n ;\ 3m-n-l ;\ 3n-l-m.$

[Besant and Ramsey : *Hydro-Mechanics*, Part I, p. 58, Q. 46.]

We can easily show that the areal co-ordinates of the centre of mass for multiples l, m, n at the angular points of a triangle are proportional to l, m, n.

Now the centre of pressure P of ABC is the centroid of masses $q+r, r+p, p+q$ at A′, B′, C′ the mid-points of the sides : [*Vide*: Greaves : *Hydrostatics*, article 47.]

Let $(x'y'z')$ and (xyz) be the co-ordinates of the centre of pressure referred to Δ^s A′B′C′ and ABC. Then from the above property of the centre of mass, we see that

$$x' : y' : z' = q+r : r+p : p+q ;$$

i.e. $$y+z-x : z+x-y : x+y-z = q+r : r+p : p+q.$$

From this we easily get

$$p : q : r = 3x-(y+z) : 3y-(z+x) : 3z-(x+y).$$

i.e, $$p : q : r = 3l-m-n : 3m-n-l : 3n-l-m,$$

Since the symmedian point is the mean centre for masses a^2, b^2, c^2 at A,B,C,

$$l : m : n = a^2 : b^2 : c^2.$$

$$\therefore \quad p : q : r = 3a^2 - b^2 - c^2 : 3b^2 - c^2 - a^2 : 3c^2 - a^2 - b^2.$$

Question 589.

(S. KRISHNASWAMI AIYANGAR):—ABCD is a quadrilateral and its diagonals meet at right angles at S. With S as focus two conics α, β are described one circumscribing the quadrilateral and the other inscribed in it. Shew that

(1) the two conics have one directrix identical;

(2) the envelope of the polars with respect to β of points on α is a conic having one focus and one directrix common with those of α, β;

(3) the lines joining the points of contact of β with the sides of the quadrilateral meet two by two at the vertices of the harmonic triangle of the quadrilateral.

Solution by L. N. Subramaniam, M.A., L. T.

Reciprocate α, β with respect to their common focus S. The quadrilateral ABCD reciprocates into a rectangle *abcd*, since the diagonals of ABCD intersect at right angles at the centre of reciprocation. Also α, β reciprocate into α', β' respectively inscribed and circumscribed in the rectangle *abcd*, β' being a circle; so that, β' is the director circle of the conic α' and is concentric with it.

Now, the common centre of α', β' corresponds to the S-directrix of α, β and therefore we see that the conics α, β have a common directrix; which is result (1)

Since the locus of the pole of a tangent to α' with respect to β' is evidently a concentric ellipse, we have the reciprocal theorem (2), *viz*: the envelope of the polars with respect to β of points on α is a conic with S as focus, and the same directrix as α and β.

Lastly, since the rectangle *abcd* is inscribed in β', result (3) follows at once by reciprocation. It is, in fact, the well-known harmonic property of a circumscribed quadrilateral.

Question 592.

(S. NARAYANA AIYAR, M.A.,) :—Show that a^n is equal to

$$\begin{vmatrix} 1, & 0, & 0, & \dots & 0, & 0 & 1 \\ 1, & 1, & 0, & \dots & 0 & 0 & 1+a \\ 1, & 2, & 1, & \dots & 0 & 0 & (1+a)^2 \\ 1, & {}_{n-1}C_1 & {}_{n-1}C_2 & \dots & n-1, & 1, & (1+a)^{n-1} \\ 1 & {}_nC_1 & nc_2 & \dots & nc_2 & nc_1 & (1+a)^n. \end{vmatrix}$$

Solution (1) *by T. P. Trivedi, M.A., LL.B., D. Krishnamurti, K. B. Madhava and N. Sankara Aiyar, M.A.*; (2) *by A. Narasinga Rao and N. B. Pendse.*

(1) Denote the determinant by D_{n+1}; subtracting each row from the next,

$$D_{n+1}=\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & a \\ 0 & 1 & 1 & \dots & 0, & a(1+a) \\ 0 & 1 & {}_{n-2}C_1 & \dots & 1 & a(1+a)^{n-2} \\ 0 & 1 & {}_{n-1}C_1 & \dots & {}_{n-1}C_1 & a(1+a)^{n-1} \end{vmatrix}$$

which is clearly the same as the determinant formed by deleting the first column and row; thus

$$D_{n+1}=a \cdot D_n.$$

By the repeated application of the above, we get

$$D_{n+1}=a^{n-1}.D_2$$

But $$D_2=\begin{vmatrix} 1, & 1 \\ 1, & 1+a \end{vmatrix}=a$$

Hence the result.

(2) Eliminate $a^0, a^1, a^2, \dots a^{n-1}$ from the $(n+1)$ identities

$$1 \quad =1$$
$$1+a \ =1+a$$
$$(1+a)^2=1+2a+a^2$$
$$(1+a)^n-a^n=1+n^{c}_{1}\,a+n^{c}_{2}\,a^2+\dots\dots\dots+na^{n-1}.$$

Then $D-a^nD'=0$, where D is the given determinant and D′ the minor of the last constituent of its leading diagonal.

But obviously $D'=1.$

Hence $D=a^n.$

Question 593.

(S. NARAYANA AIYAR, M.A.,) :—Show that,

$$\frac{\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{1\cdot2}\cdot\frac{1}{a+2}+\frac{x^3}{1\cdot2\cdot3}\cdot\frac{1}{a+3}+\dots\text{to }\infty}{\frac{1}{a+1}+\frac{x}{a+2}+\frac{x^2}{1\cdot2}\cdot\frac{1}{a+3}+\frac{x^3}{1\cdot2\cdot3}\cdot\frac{1}{a+4}+\dots\text{to }\infty}\text{ is equal to}$$

$$\frac{\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}-\frac{x^3}{a(a+1)(a+2)(a+3)}+\dots\text{to }\infty}{\frac{1}{a+1}-\frac{x}{(a+1)\,(a+2)}+\frac{x^2}{(a+1)\,(a+2)\,(a+3)}-\dots\text{ to }\infty.}$$

Solution (1) *by. K.B. Madhava* ; (2) *by T.P. Trivedi M.A., LL.B., M. Bhimasena Rao and K. Appukuttan Erady, M.A.*

(1) We have by the method of partial fractions

$$\frac{x^n}{a(a+1)\dots(a+n)}=\frac{x^n}{a}\cdot\frac{1}{n!}-\frac{x^n}{(a+1)\,1!}\cdot\frac{1}{n-1!}+\frac{x^n}{(a+2)\,2!}\cdot\frac{1}{n-2!}-\dots+\frac{1}{n!}\frac{x^n}{a+n}$$

from which we have successively

$$\frac{1}{a}=\frac{1}{a}$$

$$-\frac{x}{a(a+1)}=-\frac{x}{1!}\cdot\frac{1}{a}+\frac{x}{1!\,(a+1)}$$

$$\frac{x^2}{a(a+1)(a+2)}=\frac{x^2}{2!}\cdot\frac{1}{a}-\frac{x}{1!}\cdot\frac{x}{1!\,(a+1)}+\frac{x^2}{2!\,(a+2)}$$

and so on ; so that

$$\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^2}{a(a+1)(a+2)}-\frac{x^3}{a(a+1)(a+2)(a+3)}+\dots\text{to }\infty$$

$$=e^{-x}\left[\frac{1}{a}+\frac{x}{a+1}+\frac{x^2}{2!}\cdot\frac{1}{a+2}+\frac{x^3}{3!}\cdot\frac{1}{a+3}+\dots\text{to }\infty\right].$$

In the same way, substituting $a+1$ for a throughout, we see that the denominator of the right hand expression is e^{-x} times the denominator of the left, from which the given equality follows at once.

(2) The left side

$$=\frac{\frac{1}{x^a}\int_0^x\left(x^{a-1}+\frac{x^a}{\underline{|1}}+\frac{x^{a+1}}{\underline{|2}}+\dots\text{to }\infty\right)dx}{\frac{1}{x^{a+1}}\int_0^x\left(x^a+\frac{x^{a+1}}{\underline{|1}}+\frac{x^{a+2}}{\underline{|2}}+\dots\text{to }\infty\right)dx}=\frac{x\int_0^x e^x.x^{a-1}\,dx}{\int_0^x e^x.x^a\,dx}$$

$$\frac{x\left\{\frac{x^{a}}{a}e^{x}-\frac{x^{a+1}}{a(a+1)}e^{x}+\frac{x^{a+2}}{a(a+1)(a+2)}e^{x}-\dots\right\}}{\left\{\frac{x^{a+1}}{a+1}e^{x}-\frac{x^{a+2}}{(a+1)(a+2)}e^{x}+\frac{x^{a+3}}{(a+1)(a+2)(a+3)}e^{x}-\dots\right\}}$$

$$=\frac{\frac{1}{a}-\frac{x}{a(a+1)}+\frac{x^{2}}{a(a+1)(a+2)}-\dots\text{to }\infty}{\frac{1}{a+1}-\frac{x}{(a+1)(a+2)}+\frac{x^{2}}{(a+1)(a+2)(a+3)}-\dots\text{to }\infty}$$

on applying the process of integration by parts continuously.

Question 594.

(A. A. KRISHNASAWAMI AIYANGAR, B. A.) :—Establish the following approximate formula for the length of a circular arc

$$\frac{32768R-5376Q+168H-C}{2835}.$$

where C is the chord of the whole arc,
H……………………….half the arc,
Q……………………….quarter of the arc,
R……………………….eighth of the arc.

Remarks by K. B. Madhava, T. P. Trivedi, R. J. Pocock, D. Krishnamurti, N. B. Pendse and S. V. Venkataraya Sastri, M.A., L.T.

The denominator was wrongly printed 2735.

Using Mr. R. Vythynathaswamy's "*Note on the length of the Circular Arc*" in Volume VI, No. 4 of the *J. I. M. S.*, and with the same notation,

$$s=\Theta=a_1\sin p\Theta+a_2\sin p^2\Theta+a_3\sin p^3\Theta+a_4\sin p^4\Theta.$$

If $p=\frac{1}{2}$, $n=4$, we get

$$a_1=-\frac{2}{2835},\ a_2=\frac{16}{135},\ a_3=-\frac{512}{135},\text{ and }a_4=\frac{65536}{2835}.$$

Hence the required result.

Question 596.

(N. SANKARA IYER, M.A.) :—Show that

$$\frac{1}{2}\frac{1}{2}-\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}-\dots=\log\frac{1+\sqrt{2}}{2},$$

and

$$\frac{1}{2}\frac{1}{2}+\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}+\dots=\log 2.$$

Solution by K. B. Madhava, D. Krishnamurti, T. P. Trivedi and N. B. Pendse.

Expand $\log \frac{1}{2} \{ 1+\sqrt{1+x} \}$ by Lagrange's Series, or more easily by the differential equation $\frac{dy}{dx}=-\frac{1}{2a}\left\{ (1+x)^{-\frac{1}{2}}-1 \right\}x$ in the form

$$\frac{1}{2}\,\frac{x}{2}-\frac{1\cdot3}{2\cdot4}\,\frac{x^2}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^3}{6}-\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{x^4}{8}\ldots\ldots$$

and put $x=1$ to obtain the first result. The validity of such a step which is not expressly sanctioned by the expansion of $(1+x)^{-\frac{1}{2}}$ for $x=1$ may be questioned. This is permitted by the following theorem which I quote from Bromwich's *Infinite Series,* (P. 204)

"As a special example we may take the hyper-geometric series in which $\frac{a_n}{a_{n+1}}=\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}$, so that $\mu=1+\gamma-(\alpha+\beta)$.

Thus we see that for $x=1$, (i) if the real part of $(\gamma-\alpha-\beta)$ is positive, the hypergeometric series converges absolutely, etc."

Here $\gamma=1$, $\alpha=0$, $\beta=\frac{1}{2}$ and $\gamma-\alpha-\beta=\frac{1}{2}$, which is positive.

For the second also the same remarks apply:

Since $$(1-x^2)^{-\frac{1}{2}}=1+\tfrac{1}{2}\,x^2+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot2}\,x^4\ldots\ldots$$

$$\int\frac{(1-x^2)^{-\frac{1}{2}}-1}{x}dx=\tfrac{1}{2}\,\frac{x^2}{2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot2}\cdot\frac{x^4}{4}+\frac{\frac{1\cdot3\cdot5}{2\cdot2\cdot2}}{1\cdot2\cdot3}\frac{x^6}{6}\ldots\ldots$$

which gives

$$\tfrac{1}{2}\cdot\tfrac{1}{2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot2}\tfrac{1}{4}+\ldots\ldots\ldots\ldots=\int_0^{\frac{\pi}{2}}\frac{1-\cos\theta}{\sin\theta}d\theta$$

$$=-\left[\log\cos^2\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\log 2.$$

In fact $$\frac{a_n}{a_{n+1}}=\frac{(2n+2)(2n+2)}{(2n)\,(2n+1)}.$$

Question 597.

(A. NARASINGA RAO) :—Show that the parabola is the only curve such that the triangle formed by any three points on the curve is proportional to the triangle formed by the tangents at these points.

Solution by N. Durai Rajan and N. B. Pendse.

Take a tangent and normal to the curve as axes of x and y. Let P, P′ be two adjacent points on the curve and PT, P′T′ the tangents meeting the x—axis in T, T′ and intersecting in Q.

By hypothesis

$\triangle OPP' = \triangle QTT' \times k.$

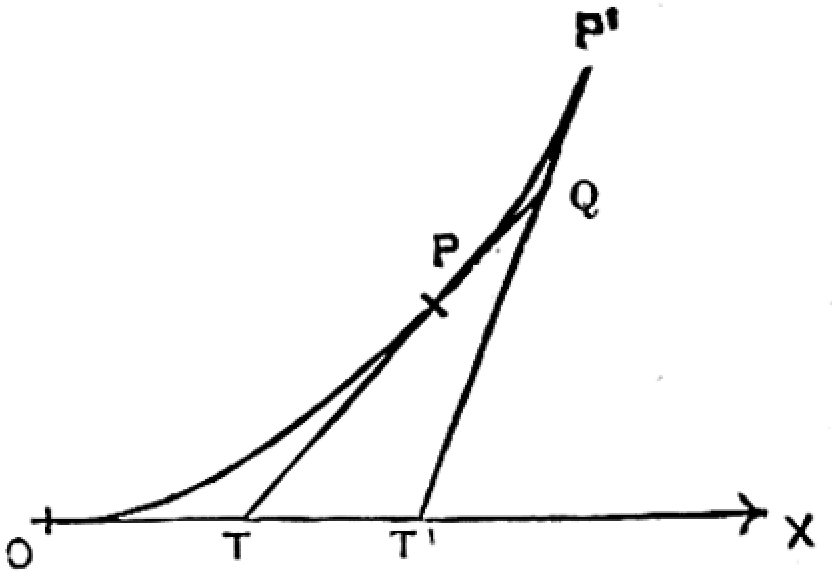
In the limit

$\triangle OPP' = \triangle PTT' \times k.$

$\therefore\ \frac{1}{2} r^2 d\theta = \frac{1}{2} k PT^2 . d\psi.$

$$\therefore\ r^2 d\theta = \frac{ky^2}{\sin^2\psi} . d\psi$$

$$= \frac{kr^2 \sin^2\theta}{\sin^2\psi} . d\psi.$$



Thus
$$\frac{d\theta}{\sin^2\theta} = k \frac{d\psi}{\sin^2\psi}.$$

Integrating
$$\cot\theta = A + k\cot\psi.$$

$$\therefore\ \frac{x}{y} = A + k\frac{dx}{dy}.$$

This is a homogeneous differential equation; by putting $x = vy$, we get for the primitive

$$Ay + x\cdot(k-1) = By^k.$$

On changing the constants

$$(x + \alpha y)^k = \beta y \quad \dots \quad \dots \quad \dots (1)$$

Take $x + \alpha y = 0$ and $y = 0$ as the new axes of y & x. The equation of the curve with then be of the form $y = a x^k$.

Let $(x_1\ y_1)$ $(x_2\ y_2)$ be two points on the curve; the tangents at these points are

$$Y - ax_1^k = a_1\cdot k\cdot x^{k-1}(X - x_1)\ ;\ etc.$$

$$\therefore\ \frac{Y}{ax_1^{k-1}} = X\cdot k - (k-1)x_1. \quad \dots \quad \dots \quad \dots (2)$$

Similarly
$$\frac{Y}{ax_2^{k-1}} = X\cdot k - (k-1)x_2. \quad \dots \quad \dots \quad \dots (3)$$

The ordinate of intersection is given by

$$\frac{Y}{a}\left(x_2^{k-1} - x_1^{k-1}\right) = (x_2x_1)^{k-1}(k-1)(x_2 - x_1).$$

The intercepted part of the x—axis between (1) and (2) is

$$\frac{k-1}{k}(x_2 - x_1).$$

By hypothesis, $\triangle OP_1P_2 = \triangle QT_1T_2 \cdot k$

$$\therefore \quad \triangle O\,P_1\,P_2 = \tfrac{1}{2}\sin w \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \tfrac{1}{2}\sin w(x_2x_1)(x_2^{k-1} - x_1^{k-1})a$$

$$= k \cdot \triangle Q\,T_1T_2 = \tfrac{1}{2}\,Y \sin w \frac{k-1}{k}(x_2 - x_1) \times k$$

$$= \tfrac{1}{2} \cdot \frac{\sin w.\ a(x_2x_1)^{k-1}(k-1)(x_2-x_1)^2}{(x_2^{k-1} - x_1^{k-1})} \cdot \frac{k-1}{k} \times k$$

$$\therefore \quad (x_2^{k-1} - x_1^{k-1})^2 = (x_2x_1)^{k-2}\,(x_2 - x_1)^2\,(k-1)^2.$$

This equation must be true for all values of x_1 and x_2.

$$\therefore \quad k = 2.$$

Hence the curve is a parabola.

Remarks. A similar problem is given in Forsyth's *Differential Equations*, Ex. 9., P. 49, 3rd Edition. The problem is: "The tangent at any point P of a curve cuts the tangent and normal at a fixed point O in the points M and N and the rectangle OMP′N is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R is equal to the △ formed by the corresponding points P′ Q′ R′."

A different method of attacking this has to be adopted.

Since the theorem is true for any three points P, Q, R, it is also true when one of the points is the origin.

Let
$$\frac{x}{a} + \frac{y}{b} = 1,\quad \frac{x}{a'} + \frac{y}{b'} = 1$$
be the tangents at Q and R. The points Q′, R′ are (a, b) and (a', b').

$$\triangle \text{ of tangents} = \tfrac{1}{2}\,(a - a')\frac{bb'}{ab' - a'b}$$

$$= \triangle\ O\ Q'\ R'$$

$$= \tfrac{1}{2}\,(ab' - a'b)$$

$$\therefore \quad bb'\,(a - a')^2 = (ab' - a'b)^2 \qquad \ldots \qquad \ldots \qquad \ldots \; (1)$$

This is true for all values of a, a', while a and b, a' and b' are connected by the same fixed relation.

From (1)
$$bb'\,(a^2 + a'^2 - 2aa') = a^2\,b'^2 + a'^2\,b^2 - 2aa'\,bb'$$

$$\therefore \quad a^2(b - b')b' = a'^2(b - b')b$$

$$\therefore \quad \frac{a^2}{b} = \frac{a'^2}{b'}.$$

Hence we see that

$$\frac{a^2}{b} = \frac{a'^2}{b'} = \frac{a''^2}{b''} = \ldots = \text{constant, } \lambda \text{ say.}$$

$$\therefore \quad a^2 = b \cdot \lambda,$$

The equation of the tangent is

$$\frac{x}{a}+\frac{\lambda y}{a^2}=1$$

or

$$a^2-ax-\lambda y=0.$$

The equation of the envelope is

$$x^2+4\lambda y=0$$

which is a parabola.

Also since

$$\triangle(123)=\triangle(012)+\triangle(023)+\triangle(031)$$

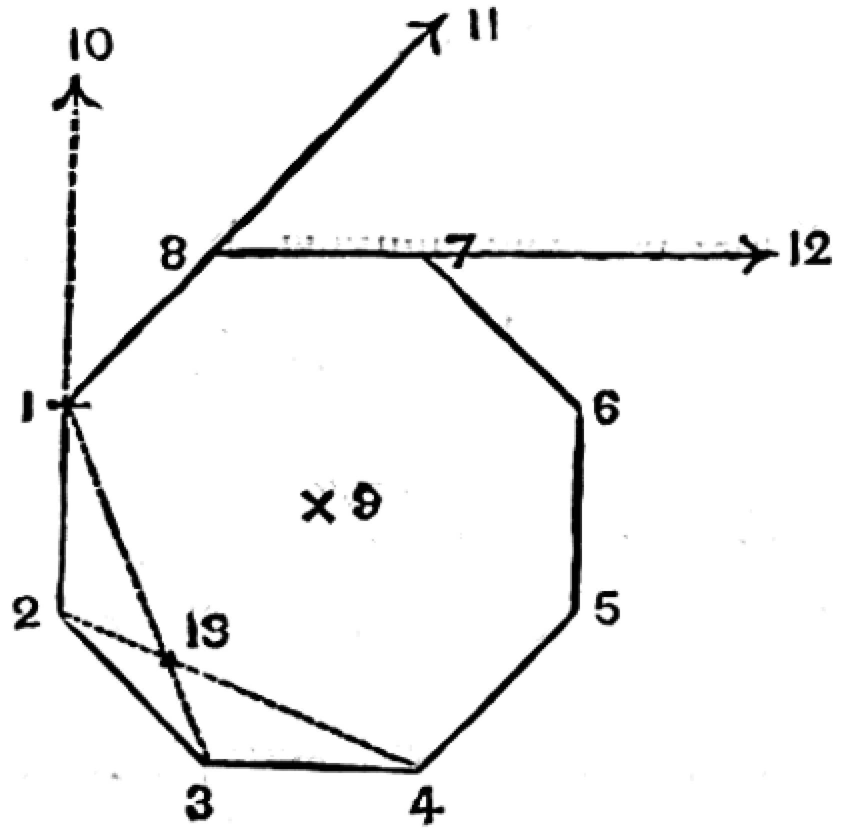
where 0, 1, 2, 3 denote either tangents or points, due regard being had to signs of areas, we see that the parabola satisfies the proposed condition generally.

Question 598.

(A. Narasinga Rao) :—Arrange 13 points so as to form 18 rows of 3 each, no row containing more than 3.

Solution (1) *by N. Durai Rajan and* (2) *by N. B. Pandse.*

Adopting a hint given in Ball's *Mathematical Recreations*, we may try the conical projection of a regular octagon.



Place one point at each of the 8 angular points,

one at a point at infinity on line 12 (point 10)

one " " 18 (" 11)

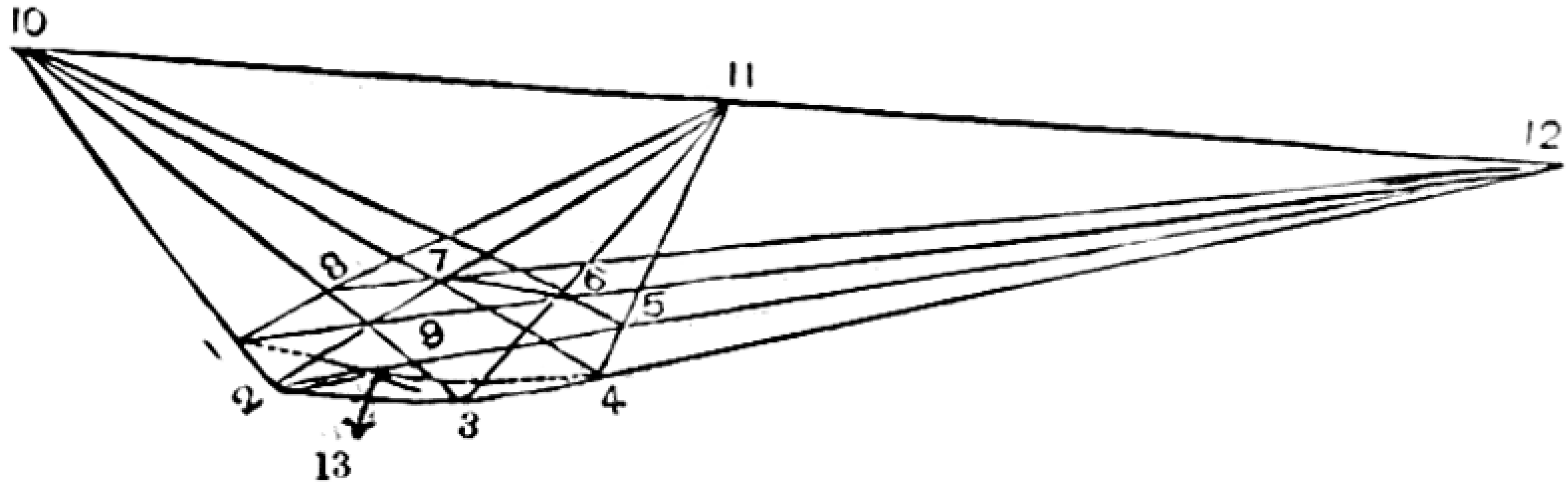
one " " 87 (" 12)

one at the centre (point 9).

one at the intersection of lines 13 and 24 (point 13).

Project the regular octagon from any vertex.

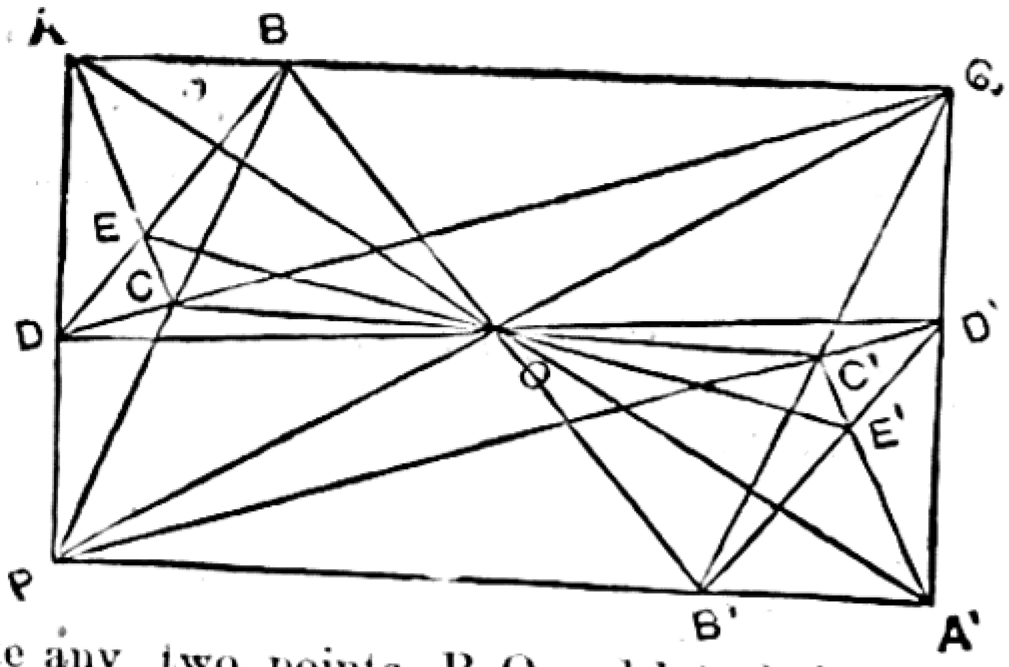
In the projected figure we have 13 points of which there are
4 rows (2, 1, 10 ; 3, 8, 10 ; 4, 7, 10 ; 5, 6, 10)
4 rows (1, 8, 11 ; 2, 7, 11 ; . . .)
4 rows (8, 7, 12 ; . . . . .)
4 rows (1, 9, 5 ; 2, 9, 6 ; 3, 9, 7 ; 4, 9, 8 ;)
1 row (10, 11, 12)
2 rows (1, 13, 3 ; 2, 13, 4.)
1 row (13, 9, 11).
since the join of 9, 13 in the regular octagon is parallel to the side (1, 8).



Thus we have 20 rows of 3 each.

One method of obtaining the projected figure is to draw the perspective representation of a horizontal regular octagon, in which points 10, 11, 12 are points on the vanishing line.

If we place 13 at any other point of line (13) or (24) we get only 18 rows.



(2) Take any two points P, Q and let their middle point be O. Draw two straight lines from P and two from Q to cut in 4 points A, B, C, D. From Q draw straight lines parallel to those at P and from P parallel to those at Q, cutting in A′ B′ C′ D′. Join A C and B D cutting at E, and A′ C′ and B′ D′ cutting at E′. It can now be shown that AA′, BB′, CC′, DD′ and EE′ are all bisected at O.

Thus there are 13 points and 18 straight lines such that not more than three are in a straight line.

Question 599.

(V. V. S. Narayan) :—If two circles cut orthogonally, find the locus of a point such that the tangents from it to the two circles form a harmonic pencil.

Solution by S. V. Venkataraya Sastri, M.A., L.T.

Let the circles be $x^2+y^2+2gx-a^2=0$, and $x^2+y^2+2fx-a^2=0$.

Since they cut orthogonally, we have $fg=a^2$.

The equation of the tangents from (x_1, y_1) to the first circle is

$$(x^2+y^2+2gx-a^2)(x_1^2+y_1^2+2gx_1-a^2)=\{ xx_1+yy_1+g\ (x+x_1)-a^2 \}^2$$

i.e.,
$$x^2(y_1^2-a^2-g^2)-2xyy_1(x_1+g)+y^2(x_1^2+2gx_1-a^2)+\ldots=0.$$

Similarly, the equation of the tangents to the second circle is

$$x^2(y_1^2-a^2-f^2)-2xyy_1(x_1+f)+y^2(x_1^2+2fx_1-a^2)+\ldots=0.$$

These four lines will form a harmonic pencil, if

$$(y_1^2-a^2-g^2)(x_1^2+2fx_1-a^2)+(x_1^2+2gx_1-a^2)(y_1^2-a^2-f^2)$$
$$=2y_1^2(x_1+g)(x_1+f),$$

which, when simplified, gives $x_1^2-a^2=0$.

Hence the required locus consists of two straight lines parallel to the common chord of the given circles and at a distance from it equal to half the common chord.

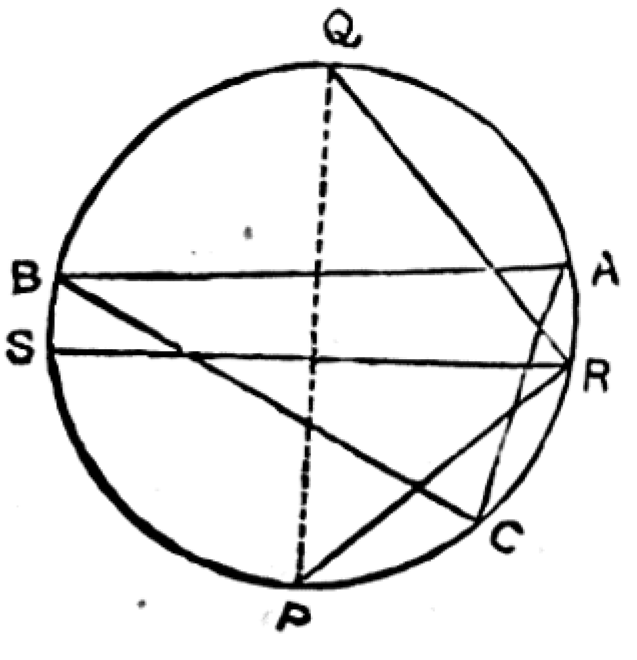
Question 600.

(V. V. S. Narayan) :—If PQ is a diameter of the circum-circle of a triangle ABC, shew that the axes of the inscribed parabolas whose foci are P and Q intersect at a point R on the circle. Shew further that the axis of the inscribed parabola whose focus is R is perpendicular to PQ.

Solution by D. Krishnamurti and N. B. Pendse.

If any point P is taken and a parabola is drawn with P as focus to touch the sides of the triangle, then the axis and AP are equally inclined to B C.

Hence the axes corresponding to two points make the same angle which the points subtend at A.



Since PQ is a diameter PAQ is a right angle. Therefore the axes are at right angles and meet on the circle.

Let them meet at R. The axis corresponding to R makes with PR an angle equal to the angle PAR, RS being that axis.

$$\angle PRS = \angle PAR.$$

$$\therefore \quad \text{arc } PS = \text{arc } PR.$$

Thus RS is at right angles to PQ.

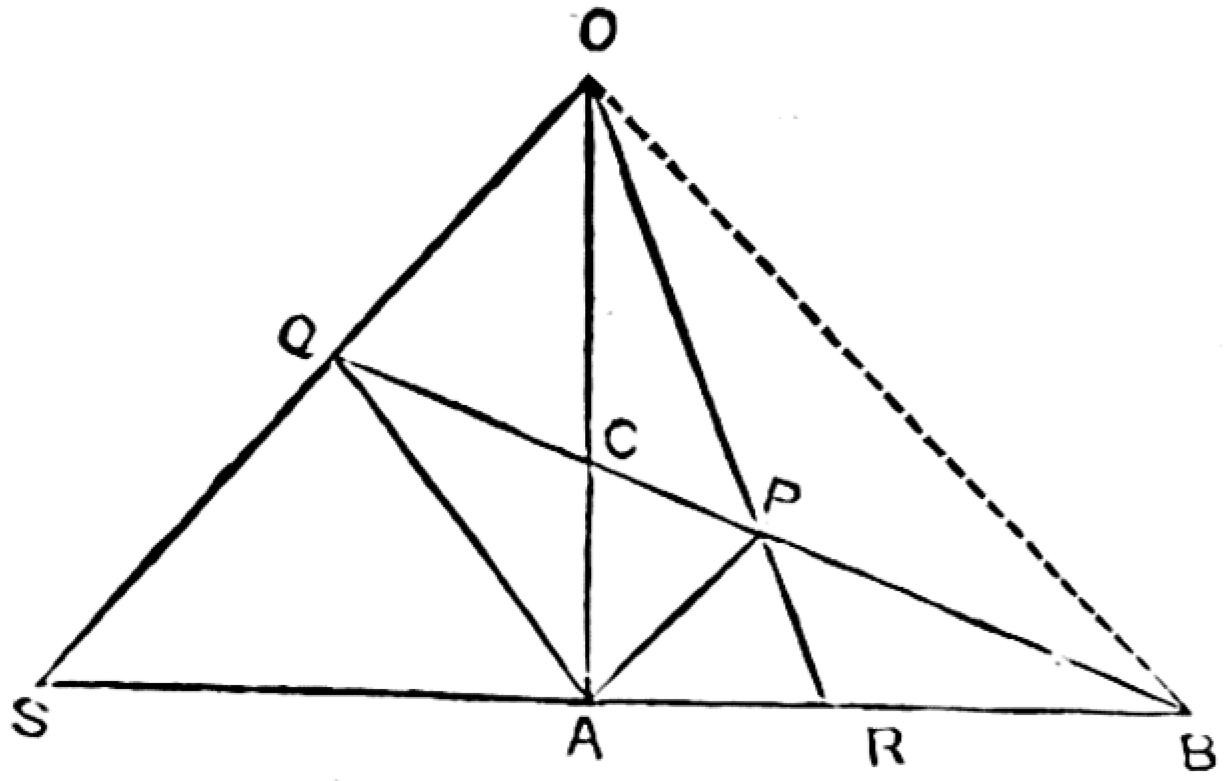
Question 602.

(S. Krishnaswami Iyengar) :—A variable straight line meets two fixed straight lines intersecting at O in the points P, Q. If OP, OQ subtend equal angles at a fixed point A, shew that PQ passes through a fixed point B such that BAO is a right angle.

Solution by V. V. S. Narayan, R. D. Karve, S. V. Venkataraya Sastri, M.A., L.T. and several others.

Let the $\perp^r$ to OA through A cut PQ, OP, OQ at B, R, S. Let PQ cut OA at C. Join OB.

AC, AB are the internal and external bisectors of the angle PAQ since, by hypothesis, OP, OQ subtend equal angles at A, and since, by construction, $C\hat{A}B$ is a right angle.



$\therefore$ { BPCQ } is a harmonic range.

$\therefore$ O { BPCQ } *i.e.*, O { BRAS } is a harmonic pencil.

$\therefore$ { BRAS } is a harmonic range, and since R, A, S, are fixed by construction, B is a fixed point.

QUESTIONS FOR SOLUTION.

668. (J. C. SWAMINARAYAN, M.A.) :—If r and n are integers, prove that the expression

$$1-\frac{2n+1}{1}{}_rC_1+\frac{(2n+1)(2n+3)}{1\cdot3}{}_rC_2-\frac{(2n+1)(2n+3)(2n+5)}{1\cdot3\cdot5}{}_rC_3+$$

$$\ldots\ldots\ldots+(-1)^r\frac{(2n+1)(2n+3)\ldots\ldots(2n+2r-1)}{1\cdot3\cdot5\cdot\ldots\ldots(2r-1)}{}_rC_r$$

is equal to $\dfrac{(-2)^r\ {}_nP_r}{1\cdot3\cdot5\ldots\ldots.(2r-1)}$, as long as r is not greater than n, but vanishes if $r>n$.

669. (J. C. SWAMINARAYAN, M. A.) :—Prove that when n is a positive integer

$$(b^2-a^2)^n+(2n+1)\frac{2n}{\underline{|2}}a^2(b^2-a^2)^{n-1}+$$

$$(2n+1)(2n+3)\frac{2n(2n-2)}{\underline{|4}}a^4(b^2-a^2)^{n-2}+\ldots\ldots\ldots\ldots+$$

$$\equiv b^{2n}+\frac{(2n)^2}{\underline{|2}}b^{2n-2}a^2+\frac{(2n)^2(2n-2)^2}{\underline{|4}}b^{2n-4}a^4+\ldots\ldots\ldots\ldots$$

$$+\frac{(2n)^2(2n-2)^2(2n-4)^2\ldots\ldots\ldots2^2}{\underline{|2n}}a^{2n}.$$

670. (K. J. SANJANA, M.A.) :—Prove that

$$\frac{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots}{\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots}=1+\frac{1}{\underline{|3}}\pi^2\frac{B_1}{2}+\frac{7}{\underline{|5}}\pi^4\frac{B_3}{2^3}+\frac{31}{\underline{|7}}\pi^6\frac{B_5}{2^5}+\ldots,$$

where B_1, B_3,...are the numbers of Bernoulli.

671. (K. J. SANJANA, M.A.) :—The numbers from 1 to 2^n-1 being arranged on n cards on which the least numbers put down are respectively 2^0, 2^1, 2^2,...2^{n-1}, as in Question 640, prove that the sum of the numbers on the rth card is $2^{2n-2}+2^{n+r-3}-2^{n-2}$.

672. (S. KRISHNASWAMI AIYANGAR) :—The vertex A of the triangle of reference lies without, on, or within the director circle of the maximum inscribed ellipse according as the angle A is acute, right, or obtuse.

673. (S. KRISHNASWAMI AIYANGAR) :—If the sides of a polygon of n sides subtend equal angles at a point S, then with S as focus two conics can be described, one circumscribing the polygon and the other inscribed in it. Show that the envelope of polars with respect to the inscribed conic of points on the circum-conic is a conic having one focus and one directrix coincident with those of the original conics.

674. (M. Bhimasena Rao) :—Evaluate

$$\int_0^\infty \left\{ \frac{\cos px \cdot \sin qx}{\cosh rx . \cos^2 qx} . \frac{dx}{x} \right\}$$

675. (K. Appukuttan Erady) :—Shew that

$$\text{(i)} \int_0^\pi \frac{e^{b \cos \theta} \cos (b \sin \theta)}{(1-2a \cos \theta + a^2)} d\theta = \frac{\pi e^{ab}}{1-a^2},$$

$$\text{(ii)} \int_0^\pi \frac{\log (1-2b \cos \theta + b^2)}{(1-2a \cos \theta + a^2)} d\theta = \frac{2\pi}{1-a^2} \log (1-ab).$$

676. (N. P. Pandya) :—Solve

$$\frac{dy^4}{dx^4} + \left(\frac{dy}{dx}\right)^4 = xy.$$

677. (D. Krishnamurti) :—A chain of length l and mass **M** is coiled at the edge of a table. A particle of mass m is fastened to one end of the chain and the other end is gently let slip over the edge of the table. Shew that the velocity of the particle immediately after it leaves the table is

$$k\sqrt{[2gl\ (1+k^2)/3]}$$

where $k = \mathrm{M}/(\mathrm{M}+m)$.

678. (D. Krishnamurti) :—Shew that the centroids of parallelograms inscribed in a skew quadrilateral, so as to have their sides parallel to the diagonals of the quadrilateral, all lie in a straight line which is also the locus of the centres of the conicoids having the sides of the quadrilateral for generators.

679. (Martyn M. Thomas) :—At each point P of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is described an ellipse having three-pointic contact with it and having a focus at the origin. Shew that the locus of the second focus is

$$\left\{ (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} \right\}^{\frac{1}{2}} \left\{ (x/a)^{\frac{2}{3}}(b^2/2 - a^2) + (y/b)^{\frac{2}{3}}(a^2/2 - b^2) \right\} = (a^2 - b^2).$$

[M. A. 1900.]

680. (R. Vythynathaswamy) :—If l, m, $n \ldots$, λ are all positive integers, find the greatest value of

$$lx + my + nz + \ldots\ldots\ldots$$

subject to the condition

$$x + y + z + \ldots\ldots = \lambda,$$

x, y, $z \ldots$ being zero or positive integers.

681. (S. RAMANUJAN) :—Solve in integers $x^3+y^3+z^3=1$, and deduce the following :

$$6^3+8^3=9^3-1,\ 9^3+10^3=12^3+1,\ 135^3+138^3=172^3-1,$$
$$791^3+812^3=1010^3-1,\ 11161^3+11468^3=14258^3+1,$$
and
$$65601^3+67402^3=83802^3+1.$$

682. (S. RAMANUJAN) :—Shew how to find the cube root of surds of the form $A+\sqrt[3]{B}$; and deduce that

$$\sqrt[3]{(\sqrt[3]{2}-1)}=\sqrt[3]{\left(\frac{1}{9}\right)}-\sqrt[3]{\left(\frac{2}{9}\right)}-\sqrt[3]{\left(\frac{4}{9}\right)}.$$

683. (GANPATRAM R. JANI) :—If S_r is put for $1^r+2^r+3^r\ldots+n^r$, prove that

$$\begin{vmatrix} S_0, & S_1, & S_2, & \ldots & \ldots & S_{n-1} \\ S_1, & S_2, & S_2, & \ldots & \ldots & S_n \\ S_2, & S_3, & S_4, & \ldots & \ldots & S_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{n-1}, & S_n, & S_{n+1}, & \ldots & \ldots & S_{2n-2} \end{vmatrix}=\left\{\frac{(n!)^n}{1^1.2^2.3^3\ldots\ldots n^n}\right\}^2.$$

The order of the determinant is given by the number of the natural numbers taken.

684. (M. BHIMASENA RAO) :—Shew that

$$\sum_{n=1}^{n=\infty}\tan^{-1}\left\{\frac{\sin(2n\pi\sqrt{1-x^2}).\,e^{-2n\pi x}}{1-\cos(2n\pi\sqrt{1-x^2})\,e^{-2n\pi x}}\right\}$$
$$=\frac{\pi}{12}\sqrt{1-x^2}-\frac{1}{4}\cos^{-1}x\,;$$

and obtain the result

$$\tan^{-1}e^{-h\pi}-\tan^{-1}e^{-3h\pi}+\tan^{-1}e^{-5h\pi}-\ldots\ldots$$
$$=\frac{1}{4}\sin^{-1}\frac{\sqrt{13}-3}{8}\ .+\frac{1}{8}\sin^{-1}\frac{\sqrt{13}-3}{4},\text{ where } h=\sqrt{13/12}.$$

685. (S. MALHARI RAO):—If n and p are any two positive integers, shew that the fraction $1/n$ may be expressed as the sum of a series of p different fractions, the numerator of each of which is 1 and the denominator a positive integer. If the least of the fractions be $\frac{(n-1)!}{(n+p-1)!}$, find the rth fraction.

List of Periodicals Received.

(From 16th May to 15th July 1915.)

1. Acta Mathematica, Vol. 37, No. 4.
2. Annals of Mathematics, Vol. 16, No. 3, March 1915.
3. Astrophysical Journal, Vol. 41, Nos. 2 and 3, March and April 1915.
4. Bulletin of the American Mathematical Society, Vol. 21, Nos. 6, 7, 8 & 9 March, April, May and June 1915, (2 Copies)
5. Bulletin des Sciences Mathematiques, Vol. 39, March, April and May 1915.
6. Educational Times, May and June 1915, (6 Copies).
7. L'Intermediaire des Mathematiciens, Vol. 22, Nos. 1, 2, 3 and 4, January, February, March and April 1915.
8. Journal de Mathematiques Elementaires, Vol. 39, Nos. 1 to 16, October 1914, to May 1915.
9. Mathematical Gazette, Vol. 8, Nos. 116 and 117, March and May 1915, (4 Copies).
10. Mathematics Teacher, Vol. 7, No. 3, March 1915.
11. Messenger of Mathematics, Vol. 44, Nos. 10 and 11, February and March 1915.
12. Monthly Notices of the Royal Astronomical Society, Vol. 75, Nos. 4 & 5, February and March 1915.
13. Philosophical Magazine, Vol. 29, Nos. 172, 173 and 174, April, May and June 1915.
14. Popular Astronomy, Vol. 23, Nos. 4, 5 and 6, April, May and June—July 1915. (3 Copies).
15. Proceedings of the Royal Society of London, Vol. 91, Nos. 628 and 629, April and May 1915.
16. Quarterly Journal of Mathematics, Vol. 46, No. 2, April 1915.
17. School Science and Mathematics, Vol. 15, Nos. 4, 5 and 6, April, May and June 1915, (3 Copies).
18. Transactions of the American Mathematical Society, Vol. 16, No. 2, April 1915.
19. Transactions of the Royal Society of London, Vol. 215, Nos. 523 to 530.
20. Tohoku Mathematical Journal, Vol. 7, Nos. 1 and 2, May 1915.
21. American Mathematical Monthly, Vol. 22, Nos. 2, 3 and 4, February, March and April 1915.
22. Rendiconti del Circolo Matematico di Palermo, Vol. 39, Nos. 1 and 2.

Note on Discontinuous flow of fluid in a Diverging Canal.

By J. M. Bose, M.A., B.Sc.

The problem of discontinuous flow of liquid in two dimensions was first successfully solved by Helmholtz (*Berlin Monatsberichte*, April 1868) and his solution of some simple cases will be found in well-known text-books on Hydrodynamics (*e.g.* Lamb, pp. 85–101) In the *Philosophical Transactions* of the Royal Soc. 1890, Michell discussed a number of cases, and all those problems which have been solved so far will be found in a memoir by Sir G. Greenhill (Advisory Committee's Report, No. 19).

One of Michell's examples in which a direct solution is given is the case of discontinuous flow of liquid in a canal of uniform section, *i.e.* with parallel sides. In the present note I propose to give an extension of this problem, *i.e.* to the discontinuous flow of fluid in a diverging canal, like a river flowing into the sea. If the river banks terminate abruptly into an ocean of infinite expanse, then the stream lines which coincide with the banks become discontinuous at this point, leaving a region of dead water behind, which plays an important part in coast erosion.

The analytical method is an application of the Schwarz-Christoffel theory of conformal representation—a detailed treatment of which will be found in text-books on the Theory of Functions and an excellent chapter on its application to Hydrodynamics will be found in Ramsey's *Hydrodynamics.*

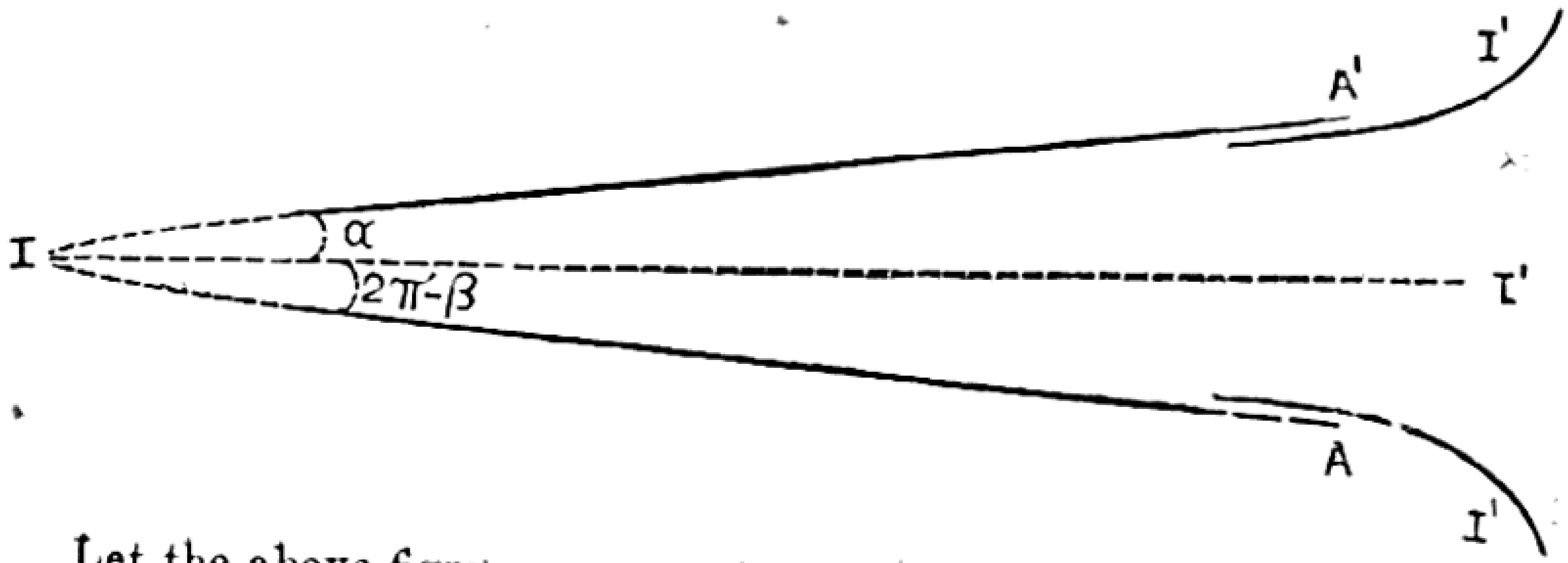
Our problem is to determine w or $\phi+i\psi$ as a function of z or $x+iy$, which satisfies certain boundary conditions. This is done by conformally transforming the curves in the $w-$plane and $z-$plane, into the upper half plane of an intermediate variable t. Schwarz-Christoffel's theorem then gives the required relation between w and z.

It is convenient however to use another notation Ω defined by

$$\Omega=\log\zeta=\log Q/q+i\theta,\ \text{and}\ \zeta=-Q\frac{dz}{dw}$$

where q and θ specify the magnitude and direction of the resultant velocity at any point.

Along plane boundaries θ is constant, and if there are no external forces then on the free surface q is constant and equal to Q, since the pressure is constant (Bernoulli's theorem). Thus the figure in the Ω-plane is bounded by straight lines.



Let the above figure represent a section of the canal (*i.e.* the z—plane) and for w, Ω and t—diagrams, we refer to Lamb's *Hydrodynamics*, p. 90.

Here, in the w—diagram, AB corresponds to $\psi=m$ and A′ B′ to $\psi=-m'$ and in the Ω-diagram AB corresponds to $\Theta=2\pi-\beta$ and A′ B′ to $\Theta=\alpha$. Let the points A, A′, I in the t—plane correspond to $t=a, a', 0$ respectively.

The $w-t$ and $\Omega-t$ relations are given by

$$\frac{dw}{dt}=\frac{N}{t}, \text{ or } w=N\log t+P;$$

$$\frac{d\Omega}{dt}=\frac{M}{t\sqrt{(t-a_1)(t-a')}}, \text{ or } \Omega=\frac{-M}{\sqrt{aa'}}.\ 2\cosh^{-1}\sqrt{\frac{a(t-a')}{(a-a')t}}+Q;$$

where $\infty>a>a'>-\infty$ in the t—plane, and P, Q, M, N are constants to be determined.

We have, when $t=a',\ \Omega=(2\pi-\beta)i;$

$t=a,\ \Omega=\alpha i.$

From these we get

$$(2\pi-\beta)i=\frac{-M}{\sqrt{aa'}}.\ 2\cosh^{-1}(0)+Q, \quad \dots \quad \dots \ (1)$$

$$\alpha i=\frac{-M}{\sqrt{aa'}}.\ 2\cosh^{-1}(1)+Q; \quad \dots \quad \dots \ (2)$$

whence

$$\frac{-M}{\sqrt{aa'}}=\frac{2\pi-\beta-\alpha}{\pi}$$

and

$$Q=\alpha i,$$

so that

$$\Omega=\frac{2\pi-(\alpha+\beta)}{\pi}.\ 2\cosh^{-1}\sqrt{\frac{a(t-a')}{(a-a')t}}+\alpha i;$$

along A′I′, $Q=q$ $\therefore$ $\Omega=\Theta i$

$$\therefore \quad \cosh\left[\frac{\pi(\Omega-\alpha i)}{2(2\pi-\alpha+\beta)}\right]=\sqrt{\frac{a(t-a')}{(a-a')t}}$$

or

$$\cos^2\left[\frac{\pi}{2}\cdot\frac{\Theta-\alpha}{2\pi-\alpha+\beta}\right]=\frac{a}{a-a'}\cdot\frac{t-a'}{t}, \quad \dots \quad \dots \ (3)$$

and $$\sin^2\left[\frac{\pi}{2}\cdot\frac{\theta-\alpha}{2\pi-\alpha+\beta}\right]=\frac{a'}{a-a'}\cdot\frac{a-t}{t}=\frac{a'}{a-a'}\left[\frac{a}{t}-1\right]. \quad \dots \quad (4)$$

Now remembering that w vanishes for $t=a$ and increases suddenly by $(m+m')i$ when t passes through the origin (*i.e.* changes sign), we have

$$w=\frac{m+m'}{\pi}\log\frac{a}{t},$$

also $$\sinh n(\Omega-\alpha i)=\sqrt{\frac{a'}{a-a'}\cdot\frac{t-a}{t}}$$

where $n=\dfrac{\pi}{2(2\pi-\alpha+\beta)}$.

Along the stream line A′I′

$$Q=q \text{ and } Q\,ds=-d\phi=-dw\,;$$

$$Qs=\frac{m+m'}{\pi}\log\frac{t}{a}, \text{ or } t=ae^{\frac{\pi s}{c}},\ m+m'=Qc.$$

Also from (4)

$$\sin^2 n\,(\theta-\alpha)=\frac{a}{a-a'}\left(\frac{a}{t}-1\right)=\frac{a'}{a-a'}\left(e^{-\frac{\pi s}{c}}-1\right)$$

giving the intrinsic equation of the free stream line.

If β be the inclination of the free stream line at a sufficient distance, *i.e.* if $\theta=\beta$ when $t=\infty$,

$$\sin^2 n(\beta-\alpha)=-\frac{a'}{a-a'}.$$

Along the bank IA′

$$\theta=\alpha$$

$$\sinh\left(n\log\frac{Q}{q}\right)=\sqrt{\frac{a'}{a-a'}\cdot\frac{t-a}{t}}=\sqrt{T} \text{ say.}$$

Thus the velocity along the bank is given by

$$\left(\frac{Q}{q}\right)^n=\sqrt{T}+\sqrt{T+1}\,;$$

or $$\frac{q}{Q}=\left[\cos n\,(\beta-\alpha)\sqrt{\frac{t-a'}{t}}-\sin n\,(\beta-\alpha)\sqrt{\frac{a-t}{t}}\right]^{\frac{1}{n}}.$$

If we put $\alpha=\beta$ and change the sign of w, we get the solution of Helmholtz's problem of Borda's re-entrant mouthpiece.

If there are a series of bends outwards, then the above method cannot be applied owing to the fact that the velocity becomes infinite at each bend. But the above method can be extended if the bends are inwards. The problem of flow along a lamina with a series of inward bends has been discussed in a paper by Bryan and Jones in the *Proc. of the Royal Soc.* (vol. 91, No. A 630.)

Squaring the Circle.*

BY S. KRISHNASWAMI AIYANGAR, B.A.

The problem of "squaring the circle" has attracted the attention of the greatest of mathematicians from the earliest times up to the present day. It is a problem whose origin is to be found in the remote antiquity and by following its history one sees how "it proceeded and changed from age to age according to the progressive development of the mathematical science." The problem, as understood by the ancient Greeks, was the determination of a square whose area is equal to that of a circle of given radius, the corners of the square being determined by constructions involving straight lines and circles only; that is, by Euclidean geometry. But the only instruments permissible in Euclidean geometry are the compasses and the ungraduated straight ruler. Hence it is that the ancient Greeks attacked the problem armed only with these weapons.

Closely allied to this problem is that of the rectification of the circle. It was known at an early date that the ratio of the circumference of a circle to its diameter is constant in value and is the same as the ratio of the area of a circle to the square on its radius. From the time of the great French mathematician Euler this constant number has been represented by the Greek letter π.

If π represented merely the ratio of the circumference of a circle to its diameter, the determination of its value would lose much of its interest. "But it really represents a certain number which would enter into analysis from whatever side the subject is approached. To illustrate how little the usual definition of π suggests its manifold properties there is an anecdote. De Morgan of whom we have heard so much, was once explaining to an actuary 'what was the chance that a certain proportion of some group of people would be alive at the end of a given time' and he quoted the actuarial formula involving π; and in answer to a query, he said π represented the ratio of the circumference of a circle to its diamerter. His friend who was until then listening to him with great interest interrupted him and exclaimed 'my dear friend, that must be a delusion; what can a circle have to do with the number of people alive at the end of a given time.'

The history of our problem embraces three distinct periods. The first period begins with the empirical determination of the ratio of the circumference to its diameter and ends in the middle of the seventeenth century when the calculus was invented. This period was mainly geometrical and mathematicians tried to determine π by calculating the sides or areas of regular polygons inscribed or circumscribed to a circle and considering that the circumference or area of the circle always lay between these values. The closeness of the approximation obtained during this period, when arithmetical notation had not reached any degree of perfection, is truly surprising. At the end of this period the geometrical method had been perfected to such a high degree that no further progress could be hoped for on the lines laid down by the Greek mathematicians.

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* **From a paper read before the Mathematics Association, Teachers' College, Saidapet, in 1914.**

The second period commenced in the middle of the seventeenth century and lasted for about a hundred years. During this period the powerful analytical methods provided by the help of the calculus were successfully employed and we find an abundance of convergent series products and continued fractions to represent π, by means of which the value of the number was determined to a larger number of decimal places than was possible before. But the real nature of the number π was still unknown and the mathematicians of this period could not say whether the number was rational or irrational.

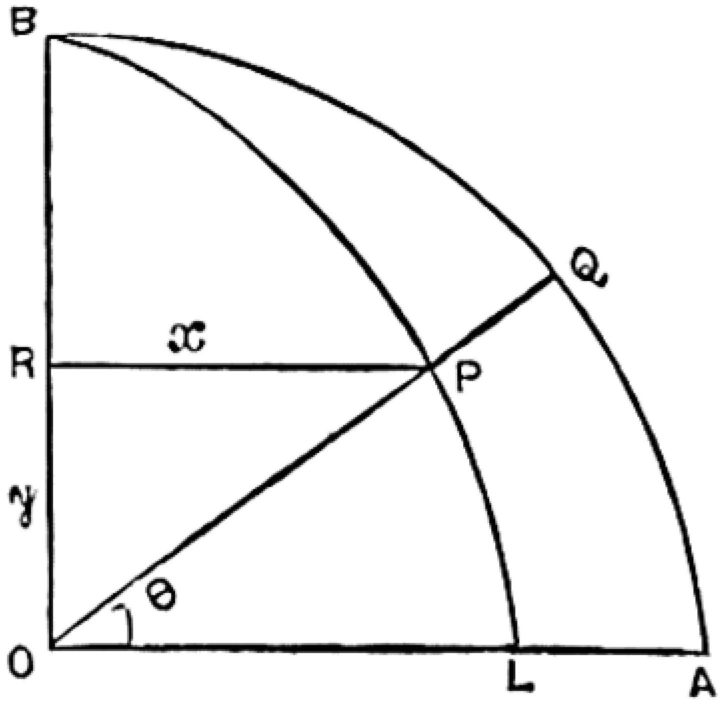
In the third period which lasted from the middle of the eighteenth century to the close of the nineteenth century the number was put to a very critical examination, and at last the discovery was made that the number did not belong to the category of 'algebraic numbers'. That is, it could not be a root of an algebraic equation with rational coefficients; and that it belonged to a class of numbers commonly known as transcendental. Thus it was conclusively proved that the number π could not be constructed by the use of Euclidean instruments or even by the use of higher algebraic curves besides the straight line and the circle.

Having thus briefly set forth the chief characteristics of the different periods, into which the history of the determination of π naturally falls, let us next examine at some length how the problem progressed according to the improvement of the general mathematical science in each period.

The earliest traces of the evaluation of π are to be found in the writings of the ancient Egyptians who used the formula: Area of a circle $=\left(\frac{8}{9}\right)^2 d^2$, which gives 3·1604...for the value of π. They also used the less accurate formula $\pi=3$.

After the ancient Egyptians came the early Greek mathematicians, who were the first to deal systematically with the problems of the rectification and quadrature of the circle. About the latter half of the fourth century B.C. Dinostratus invented and closely studied a curve called the Quadratrix by means of which he was able to find an approximate construction for π. This curve may be described as follows :—

AOB is a circular quadrant. Two points Q, R starting from A, O respectively travel along the arc AB and the radius OB and reach B at the same time. If from R a perpendicular is drawn to meet OQ at P then the locus of P will be the quadratrix. From the figure $y \propto \theta$ and $y = a (= \text{OB})$ when $\theta = \frac{\pi}{2}$.



$$\therefore \qquad y = \frac{2a}{\pi}\theta.$$

From $\triangle$OPR, $x = y \cot \theta = y \cot \frac{\pi y}{2a}$.

$$\therefore \qquad \text{OL} = \underset{y=0}{\text{L}t}\ y \cot \frac{\pi y}{2a} = \frac{2a}{\pi}.$$

Here the curve cuts the x—axis at a distance represented by $\frac{2a}{\pi}$ and hence an approximate geometrical construction of π is effected. But the construction of the curve involves the same difficulty as that of π. But a rigorous mathematical treatment of the subject begins with Archimedes who proved that π is less than $3\frac{1}{7}$ but greater than $3\frac{70}{71}$. He was able to effect this by inscribing in a circle and circumscribing about it regular polygons of 96 sides and assuming that the circumference of the circle must lie between the perimeters of these polygons. The essential point of this method translated into our modern notation becomes $\sin\theta < \theta < \tan\theta$, when $\theta = \frac{\pi}{96}$. This process is closely allied to our modern method of limits and it is in this that its chief interest lies. "But the immature condition of arithmetic at the time was the real obstacle preventing the evaluation of the ratio to any degree of accuracy whatever."

Before going to the European mathematicians of the fifteenth and sixteenth centuries let us see how the problem was developed in India and other eastern countries. Aryabhatta, the earliest of the Indian mathematicians, used the relation $\pi = \frac{62832}{20000}$, which is equivalent to 3·1416. He proved that if a_n be the side of a regular polygon of n sides inscribed in a circle of unit diameter, and Q_n that of a regular inscribed polygon of $2n$ sides then $Q_{2n} = \frac{1}{2} - \frac{1}{2}(1-a_n^2)^{\frac{1}{2}}$. From the side of an inscribed hexagon he calculated the sides of polygons of 12, 24, 48, 96, 192 and 384 sides, the last result of which becomes $\sqrt{9{\cdot}8694}$, which gives the approximation quoted above. Of the other Hindu mathematicians, Brahmagupta used the approximation $\sqrt{10}$ and Bhaskaracharya 3·1416. Approximate values were used also by the Arabs and the Chinese. But the Japanese approximations were closer than the Arabs'.

Returning now to the European mathematicians, we find that after the time of the Renaissance Adriaen Anthonisz calculated that $\pi > \frac{333}{106}$ and $< \frac{377}{120}$. He boldly asserted that its true value is equal to the mean of the numerators divided by the mean of the denominators; that is, $\frac{355}{113}$, which is a very close approximation for all practical purposes. After him came Vieta, who, by a consideration of the areas of regular polygons of n sides and $2n$ sides inscribed in a circle of unit diameter, got an infinite sequence for the value of π. He proved that

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\,\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\,\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots\cdots$$

One writer remarks that "from this period onwards, therefore, no knowledge whatever of geometry was necessary, to any one who aspired to determine the ratio to any required degree of accuracy; the mere arithmetician's art and length of days were the only requisites."

Thus a genus of workers became possible who may be styled 'π-computers', a name which if it connotes anything uncomplimentary does so because of the almost entirely fruitless character of the labours."

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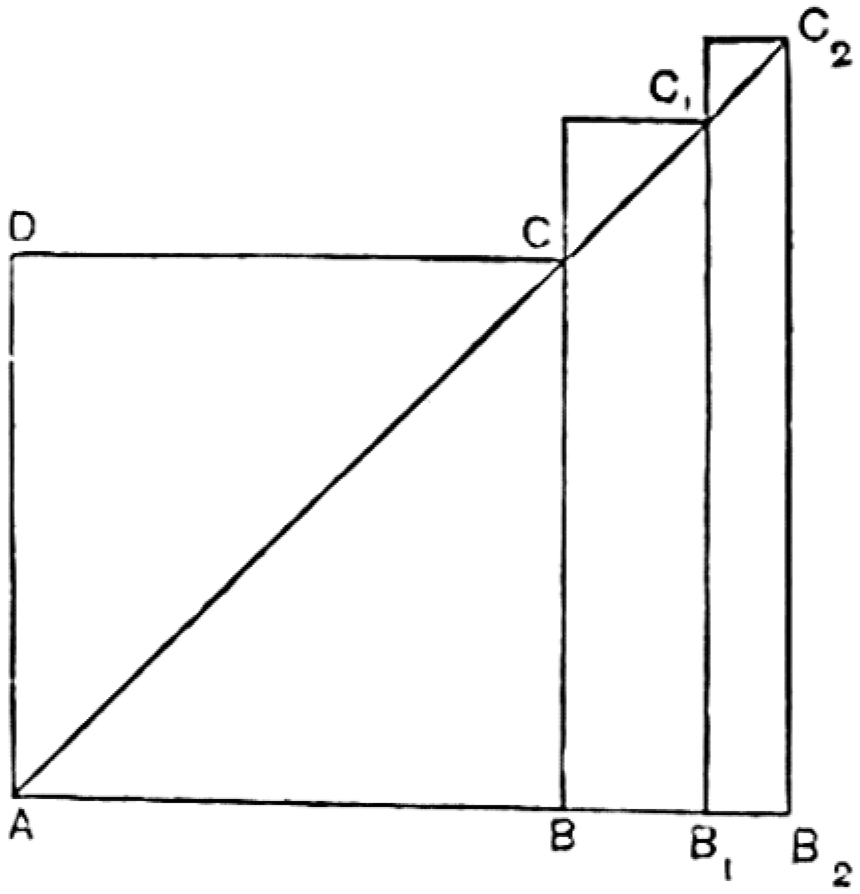
* "Squaring the Circle" by E. W. Hobson, (1913) p. 31.

In connection with the development of the method of Archimedes we have to introduce two more names of importance: Huyghens and Gregory. Gregory attempted a proof of the impossibility of the quadrature by Archimedian methods, and Huyghens reputed Gregory's proof. "Huyghens expressed his own conviction of the impossibility of the quadrature, and in his controversy with Wallis remarked that it was not even decided whether the area of a circle and the square of the diameter are commensurable or not."



But Descartes the inventor of analytical geometry approached the problem from quite a new standpoint. He started with a given straight line equal to the circumference and tried to find its diameter by the following method. On $AB=\frac{1}{4}$ of the circumference a square ABCD is constructed; on AC produced a point C_1 is taken such that rect. $BC_1=\frac{1}{4}$ sq. ABCD; similarly C_2 is taken such that rect. $B_1\ C_2=\frac{1}{4}$ rect. BC_1; and so on, *ad infinitum.* Then the diameter is given by Lt. AB_n. By the construction it is easily seen that $AB_n.\ (AB_n-AB_{n-1})=\frac{1}{4}\ AB$, which is satisfied by $AB_n=\frac{4AB}{2^n}\cot\frac{\pi}{2^n}$.

$$\therefore\quad \underset{n\to\infty}{\text{Lt.}}\ AB_n=\frac{4AB}{\cdot}\ \underset{n\to\infty}{\text{Lt.}}\ \cos\frac{\pi}{2^n}\left[\frac{\pi}{2^n}\Big/\sin\frac{\pi}{2^n}\right]=\frac{4AB}{\pi}$$

The invention of logarithms by John Napier early in the seventeenth century greatly enhanced the mathematician's attention to the number π; thence forward we find the two numbers e and π going hand in hand until at last in the 19th century these two numbers received one of the greatest developments that modern mathematics can boast of. To this we shall return later.

As has been already pointed out, the invention of the calculus in the second half of the same century led to a radical change in the ideas and methods of the mathematicians. After this date the geometrical phase of the problem was forgotten and mathematicians began to discover analytical expressions to represent π. This new method is first seen in the works of John Wallis who by an "ingenious method depending upon the expression for $\frac{\pi}{8}$ the area of a semi-circle of unit diameter as the definite integral $\int_0^1 (x-x^2)^{\frac{1}{2}}dx$" invented his well-known expression

$$\frac{\pi}{2}=\frac{2\cdot4\cdot6\cdot\ldots}{1\cdot3\cdot5\cdot\ldots}$$

But the expression most frequently used for calculating π is that given by Gregory: $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}+\ldots$ This was afterwards independently discovered by the French mathematician Leibnitz and the particular case got by substituting $x=1$ is known as Leibnitz's series. By means of Gregory's series Abraham Sharp was able to calculate the value of π to 72 decimal places. By taking the more convergent series for $\sin^{-1}x$ Newton calculated π for 14 places. In 1737 Euler employed the equation

$$\frac{\pi}{4}=\tan^{-1}\tfrac{1}{2}+\tan^{-1}\frac{1}{2\cdot4}+\tan^{-1}\frac{1}{2\cdot9}.$$

In 1746 a still more convergent series was used by Machin from which π was calculated for 100 places. "More astonishing still have been the deeds of the π computers of the 19th century." A condensed record compiled by Glaisher is given below for those interested in the subject.

Date.	Computor.	No. of digits calculated.	No. of digits correct.
1842	Rutherfield	208	152
1844	Dase	205	200
1847	Clansen	250	248
1853	Shankes	318	318
1853	Rutherfield	440	440
1853	Shanks	530	......
1853	Shanks	607	......
1853	Richter	333	330
1854	Richter	400	330
1854	Richter	400	400
1854	Richter	500	500
1873	Shanks	707	......

In this connection it may be remarked that for most mathematical calculations the value of π to 6 or 7 decimal places is quite sufficient. It has been calculated that if a circle is drawn on a distance of 200 miles as diameter and its actual circumference measured, then its value calculated by taking π to 15 decimal places will differ from the actual value by less than 18 millionth part of a millimetre. Such being the case with 15 decimal places, it need hardly be pointed out the impossibility of the human mind to conceive the degree of accuracy obtained by following the value of π for 500 or for 700 decimal places.

Before going to the third and the final period of the history of our problem let 'us see how its development underwent changes in the hands of the French mathematician Euler. It was he who first introduced the idea of ratio in place of the old sine, cosine, tangent etc. which were simply the lengths of straight lines connected with the circular arc. "Thus trigonometrical ratios became functions of an angular magnitude and therefore numbers." This mode of regarding trigonometrical ratios as analytical functions gave rise to one of the grandest discoveries in modern mathematics. Representing $1+x+\frac{x^2}{2!}+\frac{{}^3}{3!}+\dots$ by e^x he proved that cos $=\frac{e^{xi}+e^{-xi}}{2}$ and $\sin x=\frac{e^{zi}-e^{-xi}}{2i}$ which can be written in the form $e^{xi}=\cos x+i\sin x$ and $e^{-xi}=\cos x-i\sin x$. Putting $x=\pi$ in the first relation, we get $e^{\pi i}=-1$, a fundamental relation between the two numbers e and π which is of the greatest importance in proving the real nature of π. Though much improvement was made in this period nothing was yet known as to the real nature of π. In this connection it is interesting to hear what Legendre the French mathematician has to say. In his "Elements of geometry" published in the year 1794 he writes: "It is probable that the number π is not even contained among the algebraical irrationalities, that is, it cannot be a root of an algebraic equation, with a finite number of terms whose co-efficients are rational. But it seems to be very difficult to prove this strictly."

It was in the third and final period that our problem reached its highest point of development. Lambert, the German mathematician who flourished during the years 1728—77 expressed tan z in the form of an infinite continued fraction and hence proved that if x is a rational number different from zero, then tan x cannot be a rational number. If $x=\frac{\pi}{4}$ we have $\tan x=1$ and therefore $\frac{\pi}{4}$ or π cannot be a rational number.

In 1882 Lindemann, a German mathematician proved that if x_1, x_2, x_3,.........are any real or complex algebraical numbers, all distinct and p_1, p_2, p_3,.........are n algebraical numbers at least one of which is different from zero, then the sum $p_1 e^{x_1}+p_2 e^{x_2}+p_3 e^{x_3}+\dots+p_n e^{x_n}\neq 0$. Therefore as a particular case $e^{xi}+1\neq 0$. But we know that $e^{\pi i}+1=0$. Therefore π is not algebraic but transcendental.

Now, we know that every problem in pure geometry has its corresponding counter-part in analytical geometry and it can be easily proved that the determination of a geometrical construction by means of Euclidean methods corresponds, in co-ordinate geometry, to the solution of "only those algebraic equations which are obtainable by elimination from a sequence of linear and quadratic equations." But "π being transcendental, is not a root of any algebraic equation at all;" and therefore in accordance with the criterion mentioned, cannot be determined by Euclidean construction.

Thus it has been conclusively proved once for all that the "ideal of an exact construction" by means of Euclidean methods is an impossibility. In the words of Schubert "these are the words of a final determination of a controversy which is as old as the history of the human mind. But the race of circle squarers unmindful of the verdict of mathematics, the most infallible of arbiters, will never die out as long as ignorance and thirst for glory remain united."

But if an actual construction of π is desired it can only be effected by means of a "transcendental" curve. This necessitates the use of the transcendental apparatus "Integraph" which was recently invented and described by a Russian engineer. If the linkwork of the Integraph is moved in such a manner that the guiding point describes the curve $y=f(x)$, then the tracing point will trace out the integral curve $y=\int f(x)dx$. Taking for the differential curve the circle $x^2+y^2=a^2$, we see that the integral curve is

$$y=\int \sqrt{(a^2-x^2)}\,dx$$

$$=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a},$$

which cuts the y—axis in points whose ordinates are o, $\pm\, a^2\frac{\pi}{2}$, etc.

SHORT NOTES.

On the sum of the square roots of the first n Natural Numbers.

1. Let

$$\phi_1(n)=\sqrt{1}+\sqrt{2}+\sqrt{3}+\dots+\sqrt{n}-\left(C_1+\tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n}\right)$$
$$-\frac{1}{6}\sum_{\nu=0}^{\nu=\infty}\left(\sqrt{n+\nu}+\sqrt{n+\nu+1}\right)^{-3}$$

where C_1 is a constant such that $\phi_1(1)=0$. Then we see that

$$\phi_1(n)-\phi_1(n+1)=-\sqrt{n+1}+\left[\tfrac{2}{3}(n+1)\sqrt{n+1}+\tfrac{1}{2}\sqrt{n+1}\right]$$
$$-\left(\tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n}\right)+\tfrac{1}{6}\left(\sqrt{n}-\sqrt{n+1}\right)^3=0.$$

But $\phi_1(1)=0$. Hence $\phi_1(n)=0$ for all values of n. That is to say

$$(1)\quad \sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\dots+\sqrt{n}=C_1+\tfrac{2}{3}n\sqrt{n}+\tfrac{1}{2}\sqrt{n}$$
$$+\tfrac{1}{6}\left\{\left(\sqrt{n}+\sqrt{n+1}\right)^{-3}+\left(\sqrt{n+1}+\sqrt{n+2}\right)^{-3}\right.$$
$$\left.+\left(\sqrt{n+2}+\sqrt{n+3}\right)^{-3}+\dots\right\}.$$

But it is known that

$$(2)\quad C_1=-\frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\dots\dots\dots\right)$$

Putting $n=1$ in (1) and using (2), we obtain

$$(3)\quad 2\pi\left\{\frac{1}{(\sqrt{1})^3}+\frac{1}{(\sqrt{1}+\sqrt{2})^3}+\frac{1}{(\sqrt{2}+\sqrt{3})^3}+\frac{1}{(\sqrt{3}+\sqrt{4})^3}+\dots\right\}$$
$$=3\left\{\frac{1}{(\sqrt{1})^3}+\frac{1}{(\sqrt{2})^3}+\frac{1}{(\sqrt{3})^3}+\frac{1}{(\sqrt{4})^3}+\dots\right\}$$

2. Again let

$$\phi_2(n)=1\sqrt{1}+2\sqrt{2}\dots+n\sqrt{n}-\left(C_2+\tfrac{2}{5}n^2\sqrt{n}+\tfrac{1}{2}n\sqrt{n}+\tfrac{1}{8}\sqrt{n}\right)$$
$$-\frac{1}{40}\sum_{\nu=0}^{\nu=\infty}\left[\sqrt{(n+\nu)}+\sqrt{(n+\nu+1)}\right]^{-5}$$

where C_2 is a constant such that $\phi_2(1)=0$. Then we have

$$\phi_2(n)-\phi_2(n+1)=-(n+1)\sqrt{n+1}+$$
$$\left\{\tfrac{2}{5}(n+1)^2\sqrt{n+1}+\tfrac{1}{2}(n+1)\sqrt{n+1}+\tfrac{1}{8}\sqrt{n+1}\right\}$$
$$-\left\{\tfrac{2}{5}n^2\sqrt{n}+\tfrac{1}{2}n\sqrt{n}+\tfrac{1}{8}\sqrt{n}\right\}+\tfrac{1}{40}\left(\sqrt{n}-\sqrt{n+1}\right)^5=0.$$

But $\phi_2(1)=0$. Hence $\phi_2(n)=0$. In other words

$$(4)\quad 1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\dots n\sqrt{n}=C_2+\tfrac{2}{5}n^2\sqrt{n}+\tfrac{1}{2}n\sqrt{n}+\tfrac{1}{8}\sqrt{n}$$
$$+\frac{1}{40}\left\{\left(\sqrt{n}+\sqrt{n+1}\right)^{-5}+\left(\sqrt{n+1}+\sqrt{n+2}\right)^{-5}\right.$$
$$\left.+\sqrt{n+2}+\sqrt{n+3}\right)^{-5}+\dots\right\}.$$

But it is known that

$$(5)\quad C_2=-\frac{3}{16\pi^2}\left(\frac{1}{1^2\sqrt{1}}+\frac{1}{2^2\sqrt{2}}+\frac{1}{3^2\sqrt{3}}+\dots\right).$$

It is easy to see from (4) and (5) that

$$(6)\quad 2\pi^3\left\{\frac{1}{(\sqrt{1})^5}+\frac{1}{(\sqrt{1}+\sqrt{2})^5}+\frac{1}{(\sqrt{2}+\sqrt{3})^5}+\frac{1}{(\sqrt{3}+\sqrt{4})5}+\dots\right\}$$
$$=15\left\{\frac{1}{(\sqrt{1})^6}+\frac{1}{(\sqrt{2})^6}+\frac{1}{(\sqrt{3})^5}+\frac{1}{(\sqrt{4})^6}+\dots\right\}$$

3. The corresponding results for higher powers are not so neat as the previous ones. Thus for example

$$(7)\quad 1^2\sqrt{1}+2^2\sqrt{2}+3^2\sqrt{3}+\dots+n^2\sqrt{n}=C_3+\sqrt{n}\left(\frac{2}{7}n^3+\tfrac{1}{2}n^2+\frac{5}{24}n\right)$$
$$-\frac{1}{96}\{(\sqrt{n}+\sqrt{n+1})^{-3}+(\sqrt{n+1}+\sqrt{n+2})^{-3}+\dots\}$$
$$+\frac{1}{224}\{(\sqrt{n}+\sqrt{n+1})^{-7}+(\sqrt{n+1}+\sqrt{n+2})^{-7}$$
$$+(\sqrt{n+2}+\sqrt{n+3})^{-7}+\dots\};$$

$$(8)\quad 1^3\sqrt{1}+2^3\sqrt{2}+\dots+n^3\sqrt{n}=C_4+\sqrt{n}\left(\frac{2}{9}n^4+\tfrac{1}{2}n^3+\frac{\ }{24}n^2-\frac{7}{384}\right)$$
$$-\frac{1}{192}\{(\sqrt{n}+\sqrt{n+1})^{-5}+(\sqrt{n+1}+\sqrt{n+2})^{-5}+\dots\}$$
$$+\frac{1}{1152}\{(\sqrt{n}\div\sqrt{n+1})^{-9}+(\sqrt{n+1}+\sqrt{n+2})^{-9}+\dots\};$$

and so on.

The constants C_3, C_4,...can be ascertained from the well known result, *viz. the constant in the approximate summation of the series* $1^{r-1}+2^{r-1}+3^{r-1}+\dots+n^{r-1}$ *is*

$$(9)\qquad \frac{2\Gamma(r)}{(2\pi)^r}\left(\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}+\dots\right)\cos\tfrac{1}{2}\pi r,$$

provided that the real part of r *is greater than* 1.

4. Similarly we can show, by induction, that

$$(10)\qquad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\dots+\frac{1}{\sqrt{n}}=C_0+2\sqrt{n}+\frac{1}{2\sqrt{n}}$$
$$-\tfrac{1}{2}\left\{\frac{(\sqrt{n}+\sqrt{n+1})^{-3}}{\sqrt{n(n+1)}}+\frac{(\sqrt{n+1}+\sqrt{n+2})^{-3}}{\sqrt{(n+1)(n+2)}}+\dots\dots\right\}$$

The value of C_o can be determined as follows: from (10) we have

$$(11)\quad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\dots+\frac{1}{\sqrt{(2n)}}-2\sqrt{(2n)}\to C_o,$$

as $n\to\infty$. Also

$$(12)\quad 2\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{6}}+\dots\dots+\frac{1}{\sqrt{(2n)}}\right)-2\sqrt{(2n)}\to C_o\sqrt{2}$$

as $n\to\infty$.

Now subtracting (12) from (11) we see that

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\ldots-\frac{1}{\sqrt{(2n)}}\rightarrow \mathrm{C}_o(1-\sqrt{2}),\ \text{as } n\rightarrow\infty.$$

That is to say

$$(13)\quad \mathrm{C}_o=-(1+\sqrt{2})\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\ldots\right).$$

We can also show, by induction, that

$$(14)\quad \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots+\sqrt{n}=\mathrm{C}_1+\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n}+\frac{1}{24\sqrt{n}}$$

$$-\frac{1}{24}\left\{\frac{(\sqrt{n}+\sqrt{n+1})^{-5}}{\sqrt{n(n+1)}}+\frac{(\sqrt{n+1}+\sqrt{n+2})^{-5}}{\sqrt{(n+1)(n+2)}}+\ldots\ldots\right\}$$

The asymptotic expansion of $(\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n})$ for large values of n can be shown to be

$$(15)\quad \mathrm{C}_1+\frac{2}{3}n\sqrt{n}+\frac{1}{2}\sqrt{n}+\frac{1}{\sqrt{n}}\left(\frac{1}{24}-\frac{1}{1920n^2}+\frac{1}{9216n^4}-\ldots\ldots\right)$$

by using Euler-Maclaurin-sum Formula.

S. Ramanujan.

Definition of a Plane Curve.

Since it appears from Prof. Seshu Aiyar's note in a recent number of the Journal (April, 1915), that the problem of defining a plane curve is one in which readers of the Journal take interest, the following account of a definition hitherto unpublished may prove welcome. Limitations of time and space alike prevent me from giving examples illustrating either definitions previously given, such as are to be found in Young's treatise on "*The Theory of Sets of Points*", or the definition now offered; but any reader to whom the topic is not new will find the construction of examples easy.

A point O is called a limiting point of a set if there is no circle with O for centre and with radius different from zero which does not contain a point of the set other than O. This is one form of the classical definition, which we include to make this paper self-contained. It must be realised that O may or may not belong to a set of which it is a limiting point; whether O belongs to a set is entirely irrelevant to the question as to whether O is a limiting point of the set.

A set of points is called 'limited' if it does not extend to infinity, 'complete' if it contains all its limiting points, and 'closed' if it is both limited and complete. Strictly speaking, it is only in a region which is itself complete that limited sets are to be identified with sets that do not extend to infinity, but for the purpose of defining a plane curve the identification may be made for the sake of brevity.

A set of points is said to be 'completed' if to it are added all of its limiting points which originally it does not include, and two sets are said to be 'inseparable' if the sets obtained by completing them have a common point. These definitions are applicable equally to limited sets and to sets which extend to infinity; their use with unlimited sets does not require the hypothesis of points at infinity, and a branch of a hyperbola, for example, is naturally said to be separable from each of the asymptotes.

A set of points is said to be 'connected' if in every partition of it into two sets the two components are inseparable. This definition may be compared with Jordan's definition of a 'connex' or connected set, accessible to readers of English in Young's treatise mentioned above (p. 204, Theorem 20) and in Hobson's treatise on "*The Theory of Functions of a Real Variable*" (pp. 137-138); as applied to sets which do not extend to infinity the definitions are equivalent; but a hyperbola and one of its asymptotes, for example, form a set connected in accordance with Cantor's difinition, with which Jordan compares his own, but not in accordance with the definition just given.

A set of points is said to be 'perfectly' connected if every pair of points belonging to it belongs to a closed connected set contained in the original set. This is the definition which is of fundamental importance; its uses extend far beyond the application to the definition of a curve. One example must suffice to shew its bearing: if a single point is removed from a circle, the set that remains is perfectly connected, though it is of course not complete, but if two points are removed, the incomplete set that remains is not perfectly connected but consists of two perfectly connected parts; the removal of any finite number of points or of an enumerably infinite number of points from the set composing a circle leaves a set which is connected in the sense of the definition above, as well as in Jordans sense and in Cantor's, but it is not a perfectly connected set if more points than one are taken.

The 'edge' of a set is the set composed of those points of the set which are limiting points of its complement. This is the common definition, which we repeat for the sake of readers to whom it is not familiar; the complement of a set, in the case of plane sets, is simply the set composed of all those points of the plane which do not belong to the set under consideration. To remove a trivial exceptional case we require one other elementary definition: *A 'multiple' set is a set containing more than one point.* We can now give the definition proposed for a plane curve.

A plane curve is a multiple perfectly connected plane set which coincides with its own edge. To say that a set coincides with its edge is one of the simplest ways of saying that every point of the set is a limiting point of the complement; the boundary of a set is defined as the sum of the edge of the set and the edge of the complement, and a set which coincides with its edge is a set which is contained in its boundary; a set is called *capacious* if it contains points which do not belong to the edge, and a set which coincides with its edge is a set which is nos capacious.

If a set is perfectly connected, so also is the set obtained by its completion, but a set which is not capacious may give on completion a capacious set. We can assert that if the set obtained by completing a plane curve is not capacious it is itself a plane curve, and it is found that an important classification of plane curves as we define them is into those which give capacious sets and those which give curves when they are completed.

The reader will realise that the definition of a curve given in this paper excludes the "space filling curves" of Peano and Moore. This exclusion is however no disadvantage, and is clearly inevitable in any definition of a purely descriptive type (see also Young's remarks on pp. 167-168 of the book already quoted.)

It is obvious that the method used to define a curve in a plane can be used to define a curve in any surface, the only modification necessary being in the definition of the complement of a set. But if the universe containig the sets is three-dimensional, surfaces as well as curves are multiple perfectly connected sets coinciding with their edges, and the manner in which a distinction can be made between curves and surfaces I can not attempt to describe in this paper; a full account of a series of definitions is already written, and notice of its publication shall be given in the Journal.

ERIC H. NEVILLE.

The Problem of Dirichlet.

(*In ths case of a sphere.*)

[This article is adapted from French mathematical books, viz., those of Poincare, Picard and Jordan and is given here for the benefit of those who cannot read French, in as much as Dirichlet's Problem is not adequately treated in any of the English books, at least not from a purist's stand point.]

Definition. A function V of the variables x, y, z which, with its partial derivatives of the first two orders, is continuous in a volume T bounded by a closed surface S and which satisfies in the volume T the equation of Laplace *viz.*,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

is called *a harmonic function* in the region T.

Enunciation of the Problem of Dirichlet. If we are given upon a closed surface S, a continuous succession of values, associated each with a point of the surface, does there always exist a harmonic function V continuous in the volume bounded by S and taking at the various points of S the system of values given? If so how to determine it?

This is the celebrated *interior* problem of Dirichlet which has been attacked by many eminent Mathematicians and which plays an important *rōle* in the Theory of Potentials.

As applied to 'Potentials', the problem may be enunciated thus:— "Given the Newtonian Potential upon a closed surface S which contains no attracting matter within it, find the potential at any point inside S."

There is a corresponding *exterior* problem of Dirichlet for the enunciation of which refer to any of the books named in the next paragraph.

That there is a solution to this problem has been proved by Riemann (for the proof, *vide*: Pierce's 'Newtonian Potential Function' page 104 or Thomson and Tait's 'Natural Philosophy', page 170); but this proof has been pointed out to be defective by Weierstrass (For Weierstrass's objections to Reimann's proof *vide*: Picard's '*Trait D'Analys*' vol. II, page 39.)

This problem has been solved by Neumann for any convex surface and also by Poincare (for these solutions *vide*: Poincare's '*Theorie du Potential Newtonian*); we shall however restrict ourselves in this article to the case of a sphere. Before proceeding to the actual solution we give below certain theorems which will be required in the solution.

2. (*a*) *Green's Theorem.* If U & V be two functions which, with their first derivatives are continuous in the volume T bounded by a closed surface S and which have second derivatives that are finite and integrable, then

$$(1)\quad \iint\left(\mathrm{U}\frac{d\mathrm{V}}{dn}-\mathrm{V}\frac{d\mathrm{U}}{dn}\right)d\sigma=\int\!\!\int\!\!\int\left(\mathrm{U}\nabla^2\mathrm{V}-\mathrm{V}\nabla^2\mathrm{U}\right)dxdydn.$$

the surface integral being extended to the bounding surface S and the triple integral being taken throughout the volume T. [For proof *vide* Williamson's *Integral Calculus* or Routh's *Analytical Statis,* Vol. II,] The theorem is true whatever be the number of surfaces which bound the volume; only it should be noted that the derivatives $\frac{dU}{dn}$ and $\frac{dV}{dn}$ are taken along the normal exterior to the volume considered.

Cor. 1. If U & V are harmonic in T, $\nabla^2 U = 0 = \nabla^2 V$; then the Right side of (1) vanishes, and we have

$$(2) \qquad \iint \left(U\frac{dV}{dn} - V\frac{dU}{dn} \right) d\sigma = 0.$$

Cor. 2. In particular, if $U=1$ and V is harmonic, $\frac{dU}{dn} = 0$ and (2) reduces to

$$(3) \qquad \iint \frac{dV}{dn} d\sigma = 0.$$

(*b*). If (a, b, c) be the co-ordinates of a fixed point A in the volume T bounded by S, and (x, y, z) the co-ordinates of a variable point P, then the function

$$\frac{1}{r} = \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

satisfies the equation of Laplace. [*vide*: Routh or Williamson.]

It is not however continuous in T since it becomes infinite at the point A. But it is continuous in the region T_1 obtained by taking away from T a sphere Σ of infinitely small radius ρ and having A for its centre. Applying therefore (2) of *Cor.* 1 given above to $\frac{1}{r}$ and a harmonic function V in the region T_1, we have

$$\iint\limits_{S} \left(V\frac{d\left(\frac{1}{r}\right)}{dn} - \frac{1dV}{rdn} \right) d\sigma + \iint\limits_{\Sigma} \left(V\frac{d\left(\frac{1}{r}\right)}{dn} - \frac{1dV}{rdn} \right) d\sigma = 0$$

the first integral being taken over the bounding surface S of the volume T and the second integral over the small sphere Σ described above, the differentiation $\frac{d}{dn}$ being along the normal interior to T_1.

It is easy to evaluate the second integral. For, upon the sphere Σ r has a constant value ρ, the second term of the integral is therefore equal to $-\frac{1}{\rho}\iint \frac{dV}{dn} d\sigma$ which is zero by *Cor.* 2.

Also
$$\frac{d\left(\frac{1}{r}\right)}{dn} = -\frac{1}{r^2}\frac{dr}{dn} = -\frac{1}{p^2}$$

[for the direction of r is the direction of the normal exterior to the sphere, *i.e.* interior to the volume T_1; thus dr and dn are equal and have the same sign.]

$$\therefore \qquad \iint V \frac{d\left(\frac{1}{r}\right)}{dn}\, d\sigma = -\frac{1}{\rho^2}\iint V\, d\sigma.$$

Further the sphere Σ being infinitely small, the value of V at each of the points on it differs infinitely little from the value of V at the point A. This integral will therefore have for its limiting value

$$-V(a,b,c).\frac{1}{\rho^2}\iint d\sigma = -V(a,b,c,)4\pi\,;$$

we have therefore

$$(4) \qquad 4\pi V(a,b,c) = \iint_S \left(V\frac{d\left(\frac{1}{r}\right)}{dn} - \frac{1}{r}\frac{dV}{dn}\right) d\sigma$$

$\frac{d}{dn}$ being taken along the normal interior to T.

(*c*) Suppose in particular that T is a sphere of radius R with (a, b, c) for its centre; then the formula (4) becomes, as shown above,

$$4\pi V(a,b,c) = \iint V \frac{d\left(\frac{1}{r}\right)}{dn} d\sigma,$$

$\frac{d}{dn}$ being along the normal interior to T, $\frac{d\left(\frac{1}{r}\right)}{dn}$ will, in this case, be $\frac{1}{R^2}$.

$$(5) \quad \therefore \quad 4\pi V\ (a,b,c) = \frac{1}{R^2}\iint V d\sigma.$$

(*d*) A harmonic function V continuous in a volume T cannot have either a maximum or a minimum value in the interior of the volume T. If possible let V be a maximum at the point (a, b, c) in the volume T.; describe a sphere Σ of very small radius ρ about (a, b, c) as centre; then at every point on this sphere

$$V(x, y, z.) < V(a, b, c)$$

$$\therefore \qquad \iint_\Sigma \frac{V d\sigma}{4\pi\rho} < V\ (a, b, c)\iint_\Sigma \frac{d\sigma}{4\pi\rho^2}$$

$$\therefore \qquad \frac{1}{4\pi\rho^2}\iint_\Sigma V d\sigma < V(a, b, c)$$

which is contrary to result (5) given above. Hence V cannot be a maximum; similarly it cannot be a minimum.

3. Having established these preliminary theorems we proceed to the problem of Dirichlet. Firstly we shall show that if the problem admits of a solution then it has only that one solution.

If possible let there be two solutions U and V to the problem; put $U - V = W$.

Now since U is a solution, $\nabla^2 U=0$ in the volume considered; similarly $\nabla^2 V=0$ in the same volume; also by our assumption $U=V$ on the surface S, both taking upon S the given system of values.

$\therefore$ $\nabla^2 W=0$ in the volume T and $W=0$ on the surface S.

$\therefore$ W must be identically zero in T; otherwise, it must have either a maximum or a minimum value in T which is contrary to the theorem proved in (*d*).

Hence $U\equiv V$ in the volume T. There cannot therefore be two distinct solutions of the problem of Dirichlet.

4. Thus far we have been talking of *any* closed surface S bounding a volume T. Now we pass on to the particular case of a sphere.

Assuming provisionally that there is a solution of the problem in the case of a sphere, we shall determine its form and then we shall show that the function we have obtained satisfies all the conditions of the problem.

Let R be the radius of the sphere and let (a, b, c) be the co-ordinates of a point A in its interior referred to the centre as origin and let $l\ (=\sqrt{a^2+b^2+c^2})$ be its distance from the centre; then we have, according to the formula (4) of § 2,

$$4\pi V(a, b, c)=\iint\left(V\frac{d\left(\frac{1}{r}\right)}{dn}-\frac{1}{r}\frac{dV}{dn}\right)d\sigma.$$

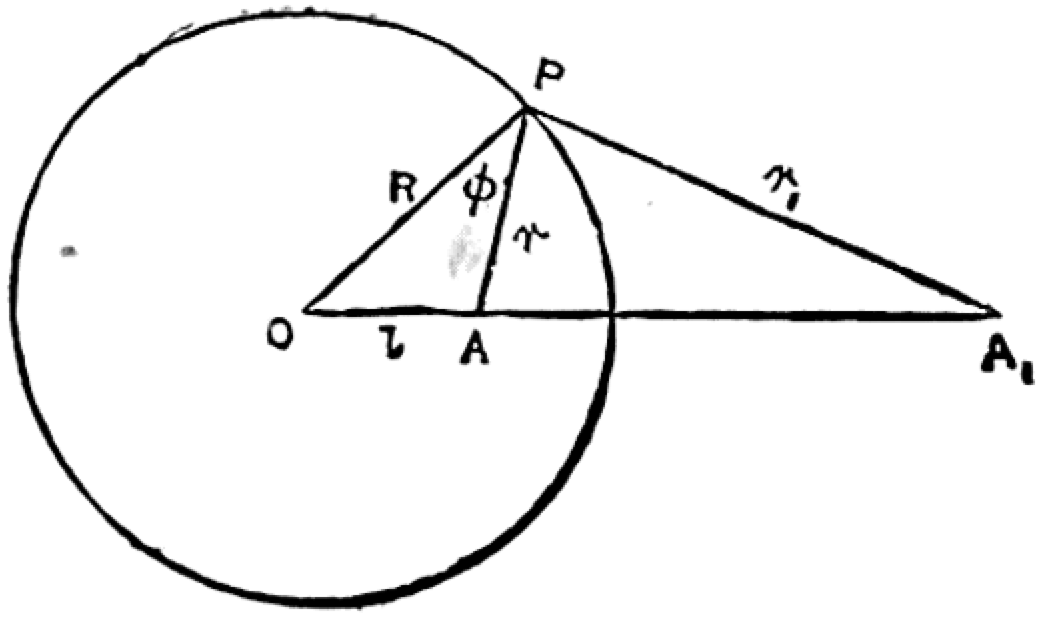
In the integral on the right hand side occurs besides V which is given on the surface, the derivative $\frac{dV}{dn}$ which is not given in the question. We use the following artifice to eliminate it:

Let A_1 be the point conjugate to the point A with respect to the sphere; then its distance l_1 from the centre is given by the formula

$$ll_1=R^2,$$

and if r, r_1 be the distances of any point on the sphere from the points A and A_1, we have

$$\frac{r}{r_1}=\frac{l}{R}$$



Now applying Green's theorem to the function V and to the function $\frac{1}{r_1}$ which is harmonic in the sphere, we have

$$\iint\left(\mathrm{V}\frac{d\left(\frac{1}{r_1}\right)}{dn}-\frac{1}{r_1}\frac{d\mathrm{V}}{dn}\right)d\sigma=0.$$

Multiply this equation by $\frac{\mathrm{R}}{l}$ and subtract it from the similar one given above; then we get

$$4\pi\mathrm{V}(a,\ b,\ c)=\iint\mathrm{V}\left(\frac{d\left(\frac{1}{r}\right)}{dn}-\frac{\mathrm{R}}{l}\,\frac{d\left(\frac{1}{r_1}\right)}{dn}\right)d\sigma\,;$$

but we have
$$\frac{d\left(\frac{1}{r}\right)}{dn}=-\frac{1}{r^2}\frac{dr}{dn}$$
$$=\frac{\cos\phi}{r^2}$$
$$=\frac{1}{r^2}\,\frac{\mathrm{R}^2+r^2-l^2}{2\mathrm{R}r}.$$

Similarly
$$\frac{d\left(\frac{1}{r_1}\right)}{dn}=\frac{1}{r_1{}^2}\,\frac{\mathrm{R}^2+r_1{}^2-l_1{}^2}{2\mathrm{R}r_1}.$$

Substituting these values and replacing r_1 and l_1 by their values $\frac{\mathrm{R}r}{l}$ and $\frac{\mathrm{R}^2}{l}$ respectively, we have

$$(6)\qquad 4\,\pi\mathrm{V}(a,b,c)=\frac{1}{\mathrm{R}}\iint\frac{\mathrm{R}^2-l^2}{r^3}\mathrm{V}\,d\sigma.$$

This is the form of the solution if there exists one. It, remains to prove that this solution satisfies the given conditions; *viz.*:

(i) V (a,b,c) is continuous and harmonic within the sphere,

(ii) that V (a,b,c) assumes at every point on the surface of the sphere the value that has been assigned to it at the point and that has been used in the calculation of the integral, *i.e.* to say, that, if $(a_o, b_o; c_o)$ be the co-ordinates of a point A_0 on the surface of the sphere and V_0 be the value assigned at A_o, V $(a,\ b,\ c)$ tends to V_0 as the point (a,b,c) tends to $(a_o,\ b_o,.c_o)$ along any path.

5. Now, so long as (a,b,c) remains in the interior of the sphere r is not zero; therefore the integral considered as a function of $(a,\ b,\ c)$ and its successive partial derivatives with respect to $(a,\ b,\ c)$ are continuous in the interior of the sphere.

Also the integral satifies the equation of Laplace, *i.e.* to say

$$\frac{\partial^2 V}{\partial a^2}+\frac{\partial^2 V}{\partial b^2}+\frac{\partial^2 V}{\partial c^2}=0.$$

For, let f denote $\frac{R^2-l^2}{r^3}$, then, remembering that $R^2=x^2+y^2+z^2$;

$$l^2=a^2+b^2+c^2\,;\ r^2=(x-a)^2+(y-b)^2+(z-c)^2$$

where x, y, z are the co-ordinates of any point on the surface of the sphere

$$\frac{\partial^2 f}{\partial a^2}=-\frac{2}{r^3}+\frac{12a(a-x)}{r^5}+(R^2-l^2)\left\{-\frac{3}{r^5}+\frac{15}{r^7}(x-a)^2\right\};$$

similarly, we get $\frac{\partial^2 f}{\partial b^2}$ and $\frac{\partial^2 f}{\partial c^2}$; and we can easily see, if we note that

$$l^2+r^2-R^2=2[a(a-x)+b(b-y)+c(c-z)],$$

that

$$\frac{\partial^2 f}{\partial a^2}+\frac{\partial^2 f}{\partial b^2}+\frac{\partial^2 f}{\partial c^2}=0.$$

$$\therefore\qquad \frac{\partial^2 V}{\partial a^2}+\frac{\partial^2 V}{\partial b^2}+\frac{\partial^2 V}{\partial c^2}=\frac{1}{4\pi R}\iint\left(\frac{\partial^2 f}{\partial a^2}+\frac{\partial^2 f}{\partial b^2}+\frac{\partial^2 f}{\partial c^2}\right)V d\sigma=0$$

i.e. $V(a, b, c)$ satisfies the equation of Laplace.

6. Lastly we have to show that

$$V(a, b, c),\ i.e.\ \frac{1}{4\pi R}\iint\frac{R^2-l^2}{r^3}V d\sigma$$

tends to V_o as (a, b, c) tends to (a_o, b_o, c_o)

Now $\frac{1}{4\pi R}\iint\frac{R^2-l^2}{r^3}\,d\sigma$ may be put equal to the sum of two integrals, viz.

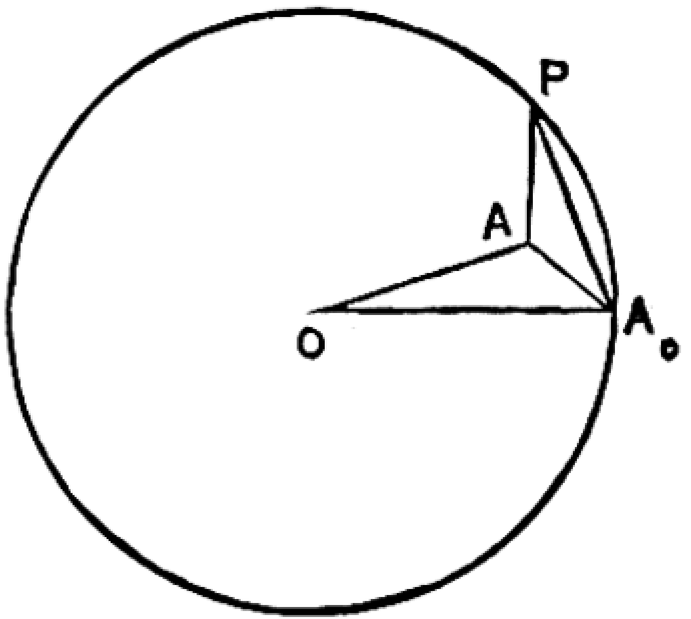
$$\frac{1}{4\pi R}\iint\frac{(R^2-l^2)}{r^3}V_o d\sigma+\frac{1}{4\pi R}\iint\frac{R^2-l^2}{r^3}(V-V_o)\ d\sigma.$$

The first integral represents a harmonic function continuous within the sphere and taking the constant value V_o on the sphere, assuming that such a function exists. It is evident that such a function exists since every constant satisfies the equation of Laplace. The first integral is thus equal to V_o. [This may also be shown by actual integration extended to the whole sphere, V_o being constant].

Therefore it remains only to prove that the second integral tends to zero.

Since the values assigned to V on the sphere vary in a continuous manner, their moduli must be less than a fixed quantity, say M. Also if we describe round the point A_o a small circle whose arcual radius subtends an angle δ at the centre, this circle divides the spherical surface into two parts; one is a small cap round A_o we shall denote it by c, and the other is the rest of the sphere, which we shall denote by C. Since V varies continuously on the sphere in the cap c, $(V-V_o)<\epsilon$ where ϵ is a small quantity which vanishes with δ.

In the rest of the spherical surface, *i.e.* in C, the distance of any point P on it from A_o will be not less than $2\,R \sin \frac{1}{2}\,\delta$. ; also if η be the distance of A from A_o, $r(=AP)$ will be not less than $A_oP - A_oA$, *i.e* not less than $2\,R \sin \frac{1}{2}\,\delta - \eta$. Lastly, $l(=OA)$ will be not less than $R-\eta$ and hence



$$R^2 - l^2 \overline{<} (2R-\eta)\,\eta.$$

Now going back to our second integral

$$\frac{1}{4\pi R}\iint \frac{R^2-l^2}{r^3}\,(V-V_0)\,d\sigma,$$

we may decompose it into two others, corresponding to the two regions c and C. Now the integral corresponding to small c, viz.

$$\frac{1}{4\pi R}\iint_c \frac{R^2-l^2}{r^3}\,(V-V_o)\,d\sigma$$

is in absolute value

$$<\frac{1}{4\pi R}\iint_c \frac{R^2-l^2}{r^3}\,\epsilon\,d\sigma.$$

i.e.
$$<\frac{\epsilon}{4\pi R}\iint_c \frac{R^2-l^2}{r^3}\,d\sigma$$

$$<\frac{\epsilon}{4\pi R}\iint_{c+C} \frac{R^2-l^2}{r^3}\,d\sigma$$

$<\epsilon$ [the integral extended to the sphere being $4\pi R$.]

Now the integral corresponding to C, viz.

$$\frac{1}{4\pi R}\iint_C \frac{R^2-l^2}{r^3}(V-V_o)\,d\sigma < \frac{1}{4\pi R}\iint_C \frac{(2R-\eta)\,\eta.\,2M.}{(2R\sin\frac{1}{2}.\delta-\eta)^3}\,d\sigma$$

$$<\frac{2M(2R-\eta)\,\eta}{4\pi R(2R\sin\frac{1}{2}\,\delta-\eta)^3}\iint_C d\sigma$$

$$<\frac{2MR.(2R-\eta)\,\eta}{(2R\sin\frac{1}{2}\,\delta-\eta)^3}.$$

This last expression tends to zero if we decrease η sufficiently rapidly compared with δ, *i.e.* so rapidly that $\frac{\eta}{R}$ is of the order δ^4 at least.

Thus both the parts into which the second integral was broken up tend to zero. Therefore the integral itself tends to zero. Therefore it follows that $V(a,\ b,\ c)$ as given by the formula (6) considered as a function of (a, b, c) is harmonic in the interior of the sphere and tends to the assigned value at any point of the surface as (a, b, c) tends to that point. Thus the interior problem of Dirichlet is completely solved in the case of a sphere.

P. V. SESHU AIYAR.

The Face of the Sky for October, November & December.

The Sun

enters Capricornus on December 23 at 3·46 A.M.

Phases of the Moon.

	October.				*November.*				*December.*			
	D.	H.	M.		D.	H.	M.		D.	H.	M.	
Last Quarter ...	1	9	44	A.M.								
New Moon ...	8	9	42	P.M.	7	7	52	P.M.	6	6	4	A.M.
First Quarter ...	15	1	52	,,	13	11	3	A.M.	13	11	38	P.M.
Full Moon ...	23	0	16	A.M.	21	5	36	,,	21	0	52	P.M.
Last Quarter ...	31	4	40	,,	29	10	11	,,	29	0	59	,,

Planets.

Mercury is an evening star in early October. Towards the end of the month in November and in the early part of December, it is a morning star. It attains its greatest elogation ($18^\circ 57'$ W) on November 7. It is in conjunction with the Moon on November 6 and December 6.

Venus is an evening star in these months. It is in conjunction with the Moon on October 9 at 8·0 P.M., on November 8 at 4 6 P.M., and on December 8 at 10·6 P.M. and with α Librae on October 27 at 1·30 A.M.

Mars is in Leo throughout these months. It is in conjunction with the Moon on October 3 at 7 11 A.M. on October 31, on November 9 at 5·47 A.M. and on December 27 at 0·38 A.M.; with Neptune on October 12 at 0·15 A.M.

Jupiter is in Pisces throughout these months. It is in retrograde motion in October and the first half of November. It is in conjunction with the Moon on October 20 at 7·55 A.M, on November 16 and on December 13 at 8·13 P.M.

Saturn is in Taurus throughout these months. It is in retrograde motion in November and December. It is in conjunction with the Moon on October 2 at 7·5 A.M. on October 29, on November 25 at 10·38 P.M. and on December 23 at 1·12 A.M.

Uranus is in conjunction with the Moon on October 17 and on December 13 at 11·13 P.M.

Neptune is in conjunction with the Moon on December 24.

V. RAMESAM.

SOLUTIONS.

Question 540.

(S. Krishnaswami Aiyangar) :—If S, H be the foci of the maximum inscribed ellipse of ABC and SH subtend angles $\theta_1, \theta_2, \theta_3$ at the vertices, prove that $bc\cos\theta_1+ca\cos\theta_2+ab\cos\theta_3=\Sigma a^2-\{\Sigma a^4-\Sigma a^2b^2\}^{\frac{1}{2}}$.

Solution by N. Sankara Aiyar, M. A.

We have $bc\cos\theta_1=3\text{AS}.\text{AH}\cos\theta_1$(J .I.M.S., Vol. V, p. 14)

$$=\tfrac{3}{2}(\text{AS}^2+\text{AH}^2-\text{SH}^2)$$
$$=\tfrac{3}{2}(2\text{AG}^2-2\text{GH}^2)$$
$$=3(\text{AG}^2-\text{GH}^2)$$
$$=3\left\{\frac{4}{9}.\frac{2(b^2+c^2)-a^2}{4}-(\alpha^2-\beta^2)\right\}$$
$$=\frac{2(b^2+c^2)-a^2}{3}-3\{\alpha^1+\beta^2)^2-4\alpha^2\beta^2\}^{\frac{1}{2}}.$$

$$\therefore\quad \Sigma bc\cos\theta_1=\sum\frac{2(b^2+o^2)-a^2}{3}-9\{(\alpha^2+\beta^2)^2-4\alpha^2\beta^2\}^{\frac{1}{2}}.$$

$$=(a^2+b^2+c^2)-\{(2\triangle\cot w)^2-12\triangle^2\}^{\frac{1}{2}}.$$
$$=(a^2+b^2+c^2)-\left\{\frac{(\Sigma a^2)^2}{4}-\frac{3}{4}16s(s-a)(s-b)(s-c)\right\}^{\frac{1}{2}}$$
$$=a^2+b^2+c^2-\left\{\frac{\Sigma a^4+2\Sigma a^2b^2}{4}-\frac{3}{4}.(2\Sigma a^2b^2-\Sigma a^4)\right\}^{\frac{1}{2}}$$
$$=a^2+b^2+c^2-\{\Sigma a^4-\Sigma a^2b^2\}^{\frac{1}{2}}.$$
$$=\Sigma a^2-\{\Sigma a^4-\Sigma a^2b^2\}^{\frac{1}{2}}.$$

Question 549.

(K. J. Sanjana, M.A.) :—If $\sum\limits_{n}^{r}=1^r-2^r+3^r+\ldots(-)^{n+1}n^r$, prove that

$$\sum_{n}^{r}+\binom{r+1}{1}\sum_{n-1}^{r}+\binom{r+2}{2}\sum_{n-2}^{r}+\ldots=0,$$

if r is even and n is any integer equal to or greater than r. Show how the value of the series may be found when r is odd.

Remarks and Solution by K. B. Madhava.

It appears to me that this is not true for finite values of n. Actual working with cases where $r=2, 4, 6$...and n equal to a small number greater than the corresponding value of r say, 8, 6..., shows that this is not true at all, except in the trivial case when $r=2$ and n is any even number.

Any meaning can therefore be attached only when in the question n is supposed to tend to infinity and with that understanding Cesaro's method of summation of Asymptotic series § can be used to get the required result.

I shall slightly change the notation for convenience into the one which is more usual, *viz.* powers of $r-1$ instead of r, with corresponding changes throughout.

$$\underset{n\to 1}{\text{Lt.}} \sum u_n x^n \text{ is defined as } \underset{n\to\infty}{\text{Lt.}} \frac{S_n^{(r)}}{A_n^{(r)}}$$

where $\overset{(r)}{\underset{n}{S}} = S_n + \binom{r}{1} S_{n-1} + \binom{r+1}{2} S_{n-2} + \dots + \binom{r+n-1}{2} S_0$ and

$$\overset{r}{\underset{n}{A}} = \frac{(r+1)(r+2)\dots.(r+n)}{n\,!}$$

But $1-2^{r-1}x^2+3^{r-1}x^3-4^{r-1}x^4+\dots$is $r-$ply indeterminate and is of the form $\frac{P(x)}{(1+x)^r}$ where P is a polynomial of degree $r-2$.

Thus

$$(1-x)^{-r-1}[(1-2^{r-1}x+3^{r-1}x^2-\dots)-l] = Q(x)(1-x^2)^{-r}$$

where Q is a polynomial of degree $r-1$ and

$$\begin{aligned} l &= 1-2^{r-1}+3^{r-1}-\dots = l \\ &= \lim_{t\to 0}\left\{ e^t - 2^{r-1}e^{2t} + 3^{r-1}e^{3t}\dots \right\} \\ &= \lim_{t\to 0} \frac{d^{r-1}}{dt^{r-1}}\left(\frac{\epsilon^t}{1+e^t}\right). \end{aligned}$$

Again expanding

$$\frac{e^t}{1+e^t} = \tfrac{1}{2} + B_1\,(2^2-1)\,\frac{t}{2\,!} - B_2(2^4-1)\frac{t^3}{4\,!} + B_3(2^6-1)\frac{t^5}{6\,!}\dots$$

so that $$\lim_{t\to 0} \frac{d^{r-1}}{dt^{r-1}}\left(\frac{e^t}{1+e^t}\right) = \left(2^{2n}-1\right)\frac{B_n}{2n} \text{ if } r=2n\,;$$

and if r is odd, $\lim \overset{r}{\underset{n}{S}} = 0$ as was required to be established.

Hence if r is even and equal to $2n$, $\lim\limits_{n\to 0} \overset{(r)}{\underset{n}{A}} = \frac{1}{r\,!}$, so that

$$\lim \overset{r}{\underset{n}{S}} = r\,!\ l = (2n-1)\,!\,(2^{2n}-1)\,B_n.$$

(§) **Bromwich §§ (122—8)**

Question 567.

(P. A. Subramanya Aiyar, B.A., L.T.) :—Show that a uniform tetrahedron of mass m is equi-momental with six particles of mass $\frac{1}{10}m$ at the middle points of its six edges and one of mass $\frac{2}{5}m$ at the centre of mass of the tetrahedron.

Solution by K. B. Madhava and D. Krishnamurti.

First, it is obvious that the two systems have the same centre of inertia.

Let OABC be the tetrahedron; P, Q, R the midpoints of OA, OB, OC; D, E, F of BC, CA, AB. Replace the particles (each of mass $\frac{1}{10}m$) at D, E and F by the equivalent particles at vertices A, B and C *viz*, by particles each of mass $\frac{1}{20}m$. Also replace those at P, Q, R in the same way. Thus we have $\frac{3}{20}m$ at O and the same amount at each of the other three vertices. The latter are again together equivalent to $\frac{9}{20}m$ at g, the centre of gravity of the triangle ABC. Obviously then the centre of gravity of the whole system is at G, the *c.g.* of the tetrahedron itself.

If we now prove that the two systems have the same moments and products of inertia about some three axes, we can show that the two systems are kinetically equivalent.

Draw through O, three straight lines OX, OY, OZ meet at right angles choosing them as co-ordinate axes. Let (x_1), $(x_2)_3$ $x_3)$ be the co-ordinates of A, B, C respectively. Then $\left(\frac{x_1}{2}\right)$; $\left(\frac{x_2}{2}\right)$; $\left(\frac{x_3}{2}\right)$ are the co-ordinates of P, Q, R and $\left(\frac{x_2+x_3}{2}\right)$, $\left(\frac{x_5+x_1}{2}\right)$, $\left(\frac{x_1+x_2}{2}\right)$ of D, E, F.

Let p be the length of the ⊥^{r} from O on the opposite face and take a section LMN parallel to ABC at distance ξ from O. Its area is $\frac{A\xi^2}{p^2}$ where A is the area of △ ABC. Now the moment of inertia of a thin slice of thickness $d\xi$ about OX

$=$ that of particles each $\frac{1}{3}A\rho\frac{\xi^2}{p^2}d\xi$ placed at the mid-points

△ LMN.

$$=\frac{1}{3}A\rho\frac{\xi^2}{p^2}d\xi\Big[\Big(\frac{\xi}{p}\frac{y_2+y_3}{2}\Big)^2+\Big(\frac{\xi}{p}.\frac{y_3+y_1}{2}\Big)^2+\Big(\frac{\xi}{p}.\frac{y_1+y_2}{2}\Big)^2$$
$$+\Big(\frac{\xi}{p}.\frac{z_2+z_3}{2}\Big)^2+\Big(\frac{\xi}{p}.\frac{z_3+z_1}{2}\Big)^2+\Big(\frac{\xi}{p}.\frac{z_1+z_2}{2}\Big)^2\Big]$$

Hence integrating *w. r. t.* ξ its moment of inertia about OX

$$=\frac{1}{5}A\rho\frac{p}{4}[\{(y_2+y_3)^2+(y_3+y_1)^2+(y_1+y_2)^2\}$$
$$+\{(z_2+z_3)^2+(z_3+z_1)^2+(z_1+z_2)^2\}].$$
$$=\frac{m}{10}[y_1^2+y_2^2+y_3^2+z_1^2+z_2^2+z_3^2+y_2y_3+y_3y_1+y_1y_2+z_2z_3+z_3z_1+z_1z_2].$$

Now, the moment of inertia about OX of six particles each of mass $\frac{m}{10}$ at P, Q, R, D, E, F and one of $\frac{2}{5}m$ at G

$$=\frac{m}{10}\Big[\Big\{\frac{y_1^2+z_1^2}{4}+\frac{y_2^2+z_2^2}{4}+\frac{y_3^2+z_3^2}{4}+\frac{(y_2+y_3)^2}{4}+$$
$$\frac{(y_3+y_1)^2}{4}+\frac{(y_1+y_2)^2}{4}+\frac{(z_2+z_3)^2}{4}+\frac{(z_3+z_1)^2}{4}+\frac{(z_1+z_2)^2}{4}\Big]$$
$$+\frac{2m}{5}\Big\{\Big(\frac{y_1+y_2+y_3}{4}\Big)^2+\Big(\frac{z_1+z_2+z_3}{4}\Big)^2\Big\}$$
$$=\frac{m}{10}[y_1^2+y_2^2+y_3^2+z_1^2+z_2^2+z_3^2+y_1y_2+y_2y_3+y_3y_1+z_1z_2+z_2z_3+z_3z_1]$$

which is the same as before.

Similarly we can show the equivalence of the moments of inertia about OY and OZ of the two systems.

In a similar manner the product of inertia of the tetrahedron about OY and OZ

$$=\frac{m}{20}[(y_2+y_3)(z_2+z_3)+(y_3+y_1)(z_3+z_1)+(y_1+y_2)(z_1+z_2)];$$

that of the seven particles

$$=\frac{m}{10}\Big[\frac{y_2z_2}{4}+\frac{y_3z_3}{4}+\frac{y_1z_1}{4}+\frac{(y_2+y_3)(z_2+z_3)}{4}+\frac{(y_3+y_1)(z_3+z_1)}{4}$$
$$+\frac{(y_1+y_2)(z_1+z_2)}{4}\Big]+\frac{2m}{5}\Big[\Big\{\frac{y_1+y_2+y_3}{4}\Big\}\Big\{\frac{z_1+z_2+z_3}{4}\Big\}$$
$$=\frac{m}{20}[(y_2+y_3)(z_2+z_3)+(z_3+z_1)(y_3+y_1)+(y_1+y_2)(z_1+z_2)],$$

which are equivalent.

Similarly, we can show the equivalence of the other two products of the two systems.

Hence the two systems are equi-momental [Routh: *Rigid Dynamics*, Vol. I. §§ 13, 15. Also Loney: *Dynamics of a Particle*, p. 198, Q. 12.]

Question 603.

(S. KRISHNASWAMI AIYANGAR) :—Show that

$$1-(x)^2+\left(\frac{x^2}{2\,!}\right)^2-\left(\frac{x^3}{3\,!}\right)^2\ldots=\frac{2}{\pi}\int_0^\pi \cos^2(x\sin\theta)d\theta-1,$$

and find the value of $1+x^2+\left(\frac{x^2}{2\,!}\right)^2+\left(\frac{x^3}{3\,!}\right)^2+\ldots\ldots$

Solution by P. V. Seshu Aiyar, K. J. Sanjana, T. P. Trivedi and N. Sankara Aiyar.

$$\text{The right hand side}=\frac{2}{\pi}\int_0^\pi \frac{1+\cos 2(x\sin\theta)}{2}d\theta-1$$

$$=\frac{2}{\pi}\int_0^\pi \left\{\frac{d\theta}{2}+\frac{\cos(2x\sin\theta)}{2}d\theta\right\}-1$$

$$=\frac{1}{\pi}\int_0^\pi \cos(2x\sin\theta)d\theta. \qquad (1)$$

Now $\cos(2x\sin\theta)=1-\frac{(2x)^2}{2\,!}\sin^2\theta+\frac{(2x)^4}{4\,!}\sin^2\theta-\quad\ldots\quad\ldots$

Hence the general term on the R. H. S. of (1) is

$$\frac{1}{\pi}\int_0^\pi (-1)^n\frac{(2x)^{2n}}{(2n)\,!}\sin^{2n}\theta d\theta$$

$$=(-1)^n\frac{(2x)^{2n}}{(2n)\,!}\,\frac{1}{\pi}\int_0^\pi \sin^{2n}\theta d\theta$$

$$=(-1)^n\frac{(2x)^{2n}}{(2n)\,!}\,\frac{1}{\pi}\cdot\frac{(2n-1)(2n-3)\ldots\ldots1}{2\,n(2n-2)\ldots\ldots\ldots\ldots2}\pi.$$

$$=(-1)^n\frac{2^{2n}x^{2n}}{2n(2n-2)\ldots2}\cdot\frac{1}{2n(2n-2)\ldots2}$$

$$=(-1)^n\frac{x^{2n}}{\{n(n-1)\ldots1)\}^2}=(-1)^n\left(\frac{x^n}{n\,!}\right)^2.$$

which is the general term on the left. Hence the given result.

By putting ix in the place of x we get

$$1+(x)^2+\left(\frac{x^2}{2\,!}\right)^2+\ldots=\frac{2}{\pi}\int_0^\pi \cosh(2x\sin\theta)d\theta.$$

Note.—The L. H. S. in the question is only $J_0\,(2x)$ and the R. H. S is the well-known integral form for J_0

Remarks by K. B. Madhava.

The first expression is the Bessel Function $J_o(2x)$ and is therefore equal to the integral on the right hand side which is more familiarly written

$$\frac{1}{\pi}\int_{o}^{\pi} \cos(2x\sin\Theta)\,d\Theta.$$

The second expression is $J_o(2ix)$, for some interesting remarks on which reference may be made to Byerly's "*Fourier's Series and Spherical Harmonics*", page 232.

Question 605.

(S. Ramanujan) :—Shew that, when $x=\infty$

$$\frac{(x+a-b)!\ (8x+2b)!\ (9x+a+b)!}{(3x+a-c!\ (3x+a-b+c)!\ (12x+3b)!}=\sqrt{\frac{2}{3}}.$$

Solution by K. J. Sanjana and A. Narasinga Rao.

By Stirling's Theorem, the limit ($x\rightarrow\infty$) of $(x+a-b)!$ is

$$\sqrt{\{2\pi(x+a-b)\}}\ .\ \left\{\frac{x+a-b}{e}\right\}^{x+a-b};$$

and similar results hold for the other factorials. Cancelling $(2\pi)^{\frac{3}{2}}$ it is seen that the limit required is the product of the limits of

$$\sqrt{\frac{(x+a-b)(8x+2b)(9x+a+b)}{(3x+a-c)(3x+a-b+c)(12x+3b)}}$$

and

$$\frac{(x+a-b)^{x+a-b}(.\quad .\quad .)(.\quad .\quad .)}{(3x+a-c)^{3x+a-c}(.\quad .\quad .)(.\quad .\quad .)}.$$

The first of these $=\sqrt{\dfrac{72+\frac{a_1}{x}+\frac{b_1}{x^2}+\frac{c_1}{x^3}}{108+\frac{a_2}{x}+\frac{b_2}{x^2}+\frac{c_2}{x^3}}}=\sqrt{\dfrac{2}{3}}.$

For the second, we see that

$$(x+a-b)^{x+a-b}=x^{x+a-b}\left\{1+\frac{a-b}{x}\right\}^{x+a-b}$$

$$=x^{x+a-b}\left\{1+\frac{a--b}{x}\right\}^{\frac{x}{a-b}\left\{a-b+\frac{(a-b)^2}{x}\right\}}$$

$$=x^{x+a-b}.e^{a-b+\frac{(a-b)^2}{x}};$$

there will be similar expressions for the other factors.

Thus the second limit

$$=\frac{x^{18x+2a+2b}.8^{8x+2b}.9^{9x+a+b}.e^{2a+2b}}{x^{18x+2a+2b}.3^{x+a-c}.3^{8x+a-b+c}.12^{12x+3b}.e^{2a+2b}}$$

$$=\frac{2^{24x+6b}.3^{18x+2a+2b}}{3^{18x+2a+2b}.2^{24x+6b}}=1.$$

Hence the given result is obtained.

Question 606.

(S. Ramanujan) :—Shew that

$$\sum_{0}^{\infty}\left\{\frac{(\sqrt{5}-2)^{2n+1}}{(2n+1)^2}\right\}=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(2+\sqrt{5})\right\}^2.$$

Solution by N. Sankara Aiyar, M.A.

Now

$$\sum_{0}^{\infty}\frac{(\sqrt{5}-2)^{2n+1}}{(2n+1)^2}=f(\sqrt{5}-2)$$

$$=\frac{\pi^2}{24}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2.\quad\text{(J. I. M. S., Vol. VII, p. 34)}$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log\left(\frac{\sqrt{5}-1}{2}\right)^3\right\}^2.$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log\frac{5\sqrt{5}-3\cdot5+3\sqrt{5}-1}{8}\right\}^2$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(\sqrt{5}-2)\right\}^2$$

$$=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log(\sqrt{5}+2)\right\}^2$$

Question 610.

(K. V. Anantanarayana Sastri, B.A.) :—If T is a point on the directrix of a parabola and t_1, t_2, the lengths of the tangents TP, TQ from it to the parabola, shew that the length of the arc PQ of the parabola is

$$\frac{t_1^3+t_2^3}{t_1^2+t_2^2}+\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{3/2}}\left\{\sinh^{-1}\frac{t_1}{t_2}+\sinh^{-1}\frac{t_2}{t_1}\right\}.$$

Solution by K. J. Sanjana, M.A. and S. Venkataraya Sastry, M.A., L.T.

With TP, TQ as axes, the equation of the curve is

$\left(\frac{x}{t_1}\right)^{\frac{1}{2}}+\left(\frac{y}{t_2}\right)^{\frac{1}{2}}=1$; and the required length of arc is

$$\int_{0}^{t_1}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx,$$

Putting $x=t_1z^2, y=t_2(1-z)^2$, $-z$ being a positive proper fraction, we get the integral

$$=\int_0^{1} \{ (t_1^2+t_2^2)z^2-2t_2^2z+t_2^2 \}^{\frac{1}{2}}dz,$$

It will be found that the indefinite integral is

$$\ddagger(z)\equiv\tfrac{1}{2}\left(z-\frac{t_2^2}{t_1^2+t_2^2}\right)\sqrt{\{ (t_1^2+t_2^2)z^2-2t_2^2z+t_2^2 \}}$$
$$+\tfrac{1}{2}\sqrt{(t_1^2+t_2^2)}\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^2}\sinh^{-1}\frac{(t_1^2+t_2^2)z-t_2}{t_1t_2},$$

omitting a constant.

Thus the length of arc is

$$2\ddagger(1)-2\ddagger(0)$$
$$=\frac{t_1^3}{t_1^2+t_2^2}+\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{3/2}}\sinh^{-1}\frac{t_1}{t_2}-\left\{\frac{-t_2^3}{t_1^2+t_2^2}+\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{7/2}}\sinh^{-1}\frac{-t_2}{t_1}\right\}$$
$$=\frac{t_1^3+t_2^3}{t_1^2+t_2^2}+\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{3/2}}\left\{\sinh^{-1}\frac{t_1}{t_2}+\sinh^{-1}\frac{t_2}{t_1}\right\}.$$

Additional Solution by L. N. Subramanian, M.A., L.T.

Question 612.

(N. Ganapathi Subba Aiyar) :—A series of ellipses have a common focus, pass through a given point and have their major axes of the same length. Find their envelope.

Solution (1) by A. Narasinga Rao, N. Durai Rajan and **N.** *B. Pendse ;*
(2) by K. J. Sanjana, M.A., T. P. Trivedi, M.A., LL.B.

(1) Let F be the common focus, P the given point and $2a$ the length of the major axis. If H is the second focus of one of the ellipses thus F P+P H$=2a$. The locus of H is a circle, centre P, and radius $r=2a-$F P.

Let H, H′ be consecutive points on this circle and Q a common point of the ellipses corresponding to H and H′, so that the locus of Q is the required envelope.

We have F Q+Q H$=2a=$F Q+Q H′. ∴ Q H$=$Q H′.

Hence Q H is ultimately normal to the circle and passes through P.
Also P Q+Q F$=r+2a=4a-$FP.

The required envelope is therefore an ellipse with F and P for foci, whose major axis is $4a-$FP.

This result may be generalised thus :—

The bipolar equation of a curve is $r'=f(r)$. If one of the poles F be fixed, the envelope of the curves which pass through a given point

P is a set of curves whose equation referred to F and P as poles is of the form $r' = f(r) + \text{const}$.

With a notation similar to the above, we have $PH = f(PF)$, and the locus of H is a circle or a series of concentric circles according as f is single or multiple-valued.

Also $$PH = f(PF),\quad QH = f(QF).$$

$$\therefore\quad PQ = f(QF) + \text{const}.$$

Hence the locus of Q is a series of curves of the form

$$r' = f(r) + \text{const}.$$

(2) Take the common focus S as origin and SP as the initial line, where P is the given common point of the ellipses.

The equation of any conic is

$$\frac{a(1-e^2)}{r} = 1 + e\cos(\theta - \alpha),$$

where a is the constant semi-major axis, e the eccentricity, and α the angle made by the axis with SP.

Let $SP = c$; since P lies on the conic

$$\frac{a(1-e^2)}{c} = 1 + e\cos\alpha$$

or $$\cos\alpha = \frac{a(1-e^2)-c}{ce}.$$

Substituting in the equation of the conic, transposing and squaring,

$$\left\{\frac{a(1-e^2)}{r} - 1 - \frac{a(1-e^2)-c}{c}\cos\theta\right\}^2$$
$$= e^2\sin^2\theta\sin^2 a = e^2\sin^2\theta\,(1-\cos^2 a)$$
$$= e^2\sin^2\theta - \left\{\frac{a(1-e^2)-c}{c}\right\}^2\sin^2\theta,$$

whence

$$(1-e^2)^2\{a^2(c^2-2cr\cos\theta+r^2)\} - (1-e^2)\{2acr(c+r)(1-\cos\theta)\}$$
$$- r^2c^2(1-\cos\theta)(1+\cos\theta) + r^2c^2(1-\cos\theta)^2 = 0.$$

Taking the discriminant of $(1-e^2)$, we get for the envelope

$$\{2a(c+r) - rc(1+\cos\theta)\}^2 = 4a^2(c^2 - 2cr\cos\theta + r^2),$$

or $$8a^2 + rc(1+\cos\theta) - 4a(c+r) = 0,$$

or $$(8a^2 - 4ac + cx)^2 = (4a-c)^2(x^2+y^2),$$

which is a conic section.

Question 615.

(Selected) :—Find two numbers such that the sum of all the aliquot parts of either is equal to the other.

Solution by **K. Ramaswami Iyengar, B.A., B.L.**

Suppose the numbers required to be $N = 2^n.p.q$ and $N' = 2^n.r$ where p, q, r are primes.

Since the sum of the aliquot parts of one is equal to the other, we have the following equations involving p, q, r.

$$(1+2+2^2+\dots 2^n)(1+p)(1+q)-\mathrm{N}=\mathrm{N}',$$

$$(1+2+\dots 2^n)(1+r)-\mathrm{N}'=\mathrm{N}.$$

$$\therefore \quad (2^{n+1}-1)(1+p)(1+q)=\mathrm{N}+\mathrm{N}'$$

$$=2^n(pq+r)=(2^{n+1}-1)(1+r).$$

Hence we get, $(p+1)(q+1)=(r+1)$ ... ... (1)

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{1}{2^n} \quad \dots \quad \dots \quad \dots \quad (2)$$

Now, to satisfy (2) put $p+1=2^n(2^k+1)$,

$q+1=2^n(2^{-k}+1)$.

Taking a very simple case, when $k=0$,

$p+1=3{\cdot}2^n$ and $q+1=3{\cdot}2^{n-1}$.

The numbers are

$$\mathrm{N}=2^n(3{\cdot}2^n-1)(3{\cdot}2^{n-1}-1)$$

and

$$\mathrm{N}'=2^n(3^2{\cdot}2^{2n-1}-1)$$

where n is so chosen as to make $3{\cdot}2^n-1$, $3{\cdot}2^{n-1}-1$, $3^2{\cdot}2^{2n-1}-1$ primes.

When $n=2$, 4 the numbers are (220, 284) and (17296, 18416).

Question 616.

(S. Narayana Aiyar, M. A.) :—Establish the following extension of Maclaurin's theorem.

$$\phi(x)+\frac{a-b}{b}\cdot\frac{x}{1\,!}\phi'(x)+\frac{(a-b)(a-b-1)}{b(b+1)}\cdot\frac{x^2}{2\,!}\phi''(x)+\dots$$

$$=\phi(0)+\frac{a}{b}\cdot\frac{x}{1\,!}\phi'(0)+\frac{a(a+1)}{b(b+1)}\cdot\frac{x^2}{2\,!}\phi''(0)+\dots$$

(a) When $a=b$, this reduces to Maclaurin's theorem.

(b) When $\phi(x)=e^x$, this reduces to Question 576 (J. I. M. S.)

(c) Examine the case when $\phi(x)=\cos x$, $(1+x)^{-n}$ *etc.*

Solution by K. B. Madhava, R. J. Pocock, T. P. Trivedi, and M. Bhimasena Rao.

This is obtained by a repeated application of Maclaurin's theorem. What we have to prove is merely

$$\frac{1}{n\,!}+\frac{a-b}{b}\cdot\frac{1}{1\,!}+\frac{(a-b)(a-b-1)}{b(b+1)}\cdot\frac{1}{2\,!}+\frac{(a-b)(a-b-1)(a-b-2)}{b(b+1)(b+2)}\cdot\frac{1}{3\,!}+\dots$$

$$=\frac{a(a+1)(a+2)\dots(a+n-1)}{b(b+1)(b+2)\dots(b+n-1)}\cdot\frac{1}{n\,!}\,;$$

and this is but a slight alteration of Mr. Sanjana's problem No. 16755, solved in the "Educational Times" Vol. 18, p. 93, with his b and c respectively $-(a-b)$ and b of this problem.

(a) Obviously it is Maclaurin's theorem when $a=b$; for, all the terms, except the first, on the L.H.S. vanish.

(b) As the proposer himself points out, if $\phi(x)=e^x$ it is his Question 576 for e^x is not altered in its form by any number of differentiations.

(c) With $\phi(x)=\cos x$, we get

$$\cos x-\frac{(a-b)}{b}\frac{x}{1!}\sin x-\frac{(a-b)(a-b-1)}{b(b+1)},\frac{x^2}{2!}\cos x+\ldots$$

$$=1.\frac{a(a+1)}{b(b+1)}\frac{x^2}{2!}+\frac{a(a+1)(a+2)(a+3)}{b(b+1)(b+2)(b+3)}\cdot\frac{x^4}{4!}-\ldots$$

as we should expect it without odd powers on the R.H.S.; it is but a most general example which would come under his formula (3) which he has obtained on p. 184 of the J. I. M. S. for October 1913.

(d) With $\phi(x)=(1+x)^{-n}$, we get

$$\frac{1}{(1+x)^n}\left\{1-\frac{a-b}{b}\cdot\frac{n}{1}\cdot\frac{x}{1+x}+\frac{(a-b)(a-b-1)n(n+1)}{b(b+1)\qquad 1\cdot2}\left(\frac{x}{1+x}\right)^2\ldots\right\}$$

$$=1-\frac{a}{b}\cdot\frac{n}{1}\cdot x+\frac{a(a+1)}{b(b+1)}\cdot\frac{n(n+1)}{1\cdot2}x^2-\ldots$$

which, reducing it to Mr. Sanjana's form, is

$$\frac{1}{(1+x)^n}\left\{1+\frac{b}{c}\cdot\frac{n}{1}\left(\frac{x}{1+x}\right)+\frac{b(b+1)}{c(c+1)}\cdot\frac{n(n+1)}{1\cdot2}\cdot\left(\frac{x}{1+x}\right)^2\ldots\right\}$$

$$=1-\frac{(c-b)}{c}\cdot\frac{n}{1}\cdot+\frac{(c-b)(c-b+1)}{c(c+1)}\cdot\frac{n(n+1)}{1\cdot2}\cdot x^2-\ldots$$

which states that

$$F(n,\ c-b,\ c,-x)=\frac{1}{(1+x)^n}F\left(n,\ b,\ c,\frac{x}{1+x}\right)$$

with the usual notation F for the hypergeometric series.

This result viz:

$$F(\alpha,\ \beta,\ \gamma,\ x)=\frac{1}{(1-x)^{\alpha}}F\left(\alpha,\ \gamma-\beta,\ \gamma,\frac{x}{x-1}\right)$$

can be more easily established by Euler's Transformation as suggested by Ex. 9. p. 170 of Bromwich.

QUESTIONS FOR SOLUTION.

686. (S. Narayana Aiyar, M. A.) :—Establish the following identities :—

i. $$\frac{a(a+1)(a+2)\ldots(a+n-1)}{b(b+1)(b+2)\ldots(b+n-1)}=1+{}_nC_1.\frac{a-b}{b}+{}_nC_2.\frac{(a-b)(a-b-1)}{b(b+1)}$$
$$+\ldots\ldots+\frac{(a-b)(a-b-1)\ldots(a-b-n+1)}{b(b+1)\ldots\ldots(b+n-1)}.$$

ii. $$\frac{(a-b)(a-b-1)(a-b-2)\ldots(a-b-n+1)}{b(b+1)(b+2)\ldots(b+n+1)}=(-1)^n.\left\{1-{}_nC_1.\frac{a}{b}\right.$$
$$\left.+{}_nC_2.\frac{a(a+1)}{b(b+1)}-\ldots(-1)^n\frac{a(a+1)\ldots(a+n-1)}{b(b+1)\ldots(b+n-1)}\right\}.$$

iii. $F(a,\ b,\ c,\ x)=(1-x)^{-a}.\ F\left(a,\ c-b,\ c,\ \frac{x}{x-1}\right)$, where F stands for the hypergeometric series.

iv. $$\left\{x.\frac{a}{b}-\frac{x^2}{2}.\frac{a(a+1)}{b(b+1)}+\frac{x^3}{3}.\frac{a(a+1)(a+2)}{b(b+1)(b+2)}-\ldots\ldots\right\}$$
$$-\left\{\frac{x}{1+x}.\frac{a-b}{b}-\frac{x^2}{2(1+x)^2}.\frac{(a-b)(a-b-1)}{b(b+1)}+\ldots\ldots\right\}=\log(1+x).$$

v. $$\pi.\frac{a}{b}-\frac{\pi^3}{3!}.\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\frac{\pi^5}{5!}.\frac{a(a+1)(a+2)(a+3)(a+4)}{b(b+1)(b+2)(b+3)(b+4)}-\ldots\ldots$$
$$=-\left\{\pi.\frac{a-b}{b}-\frac{\pi^3}{3!}.\frac{(a-b)(a-b-1)(a-b-2)}{b(b+1)(b+2)}+\ldots\ldots\right\}$$

vi. $$\frac{\pi^2}{2!}.\frac{a(a+1)}{b(b+1)}+\frac{\pi^4}{4!}\frac{a(a+1)(a+2)(a+3)}{b(b+1)(b+2)(b+3)}-\ldots\ldots$$
$$=-\left\{1-\frac{\pi^2}{2!}.\frac{(a-b)(a-b-1)}{b(b+1)}+\ldots\ldots\right\}$$

687. (K. J. Sanjana, M.A.) :—Find the envelope of the straight line $ul^2+vm^2+w+2u'm+2v'l+2w'lm=0$, where u, v, w, u', v', w' are given linear functions of the co-ordinates $(x,\ y)$ and l, m are parameters connected by the relation $pl^2+qm^2=1$.

What will be the degree of the envelope equation when u, v, w, u', v', w' are all or some of them quadratic functions?

688. (Ganpatram R. Jani) :—S_r being the sum of the r^{th} powers of the first n natural members, prove that

$$S_{r+1}=(r+1)\int^n S_r\,dn+qn.$$

When r is even, $q=0$; when r is odd, q may be found by giving numerical values to n.

689. (J. C. Swaminarayan, M.A). :—If $S_m = 1^m + 2^m + 3^m + \dots + n^m$, prove that

$$7S_6 + 5S_4 = 12S_2S_1^2.$$

690. (J. C. Swaminarayan, M.A.) :—If S_r denotes the sum of the reciprocals of the first r odd natural numbers, prove the following relations :—

$$\text{(i)}\quad \frac{S_1}{1\cdot 2} - \frac{S_2}{3\cdot 4} + \frac{S_3}{5\cdot 6} - \frac{S_4}{7\cdot 8} + \dots\dots\dots$$

$$= -\frac{\pi^2}{32} + \frac{\pi}{8}\log 2 + \tfrac{1}{2}\left\{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right\}$$

$$\text{(ii)}\quad \frac{1}{1\cdot 2} - \frac{S_1}{3\cdot 4} + \frac{S_2}{5\cdot 6} - \frac{S_3}{7\cdot 8} + \dots\dots\dots$$

$$= -\frac{\pi^2}{32} + \frac{\pi}{4} + \left(\frac{\pi}{8} - \frac{1}{2}\right)\log_t 2 + \tfrac{1}{2}\left\{\frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2} + \dots\right\}.$$

691. (A. Narasinga Rao) :—A marble slab of n pounds breaks into k pieces with which a tradesman finds himself able to weigh goods from 1 to n pounds (fractions excluded). Show that the least value of k is the smallest integer satisfying the relation $3^k \geq 2n+1$.

What are the weights of the several pieces ?

692. (R. Vythynathaswamy, B.A.) :—Prove that there cannot exist a hyperboloid, which is such that three and therefore all generators of each system are nul lines *w. r. t.* a system of forces.

A system of forces acts along generators of one system of a hyperboloid and l, m are the two generators of that system which are nul lines. Shew that the nul plane of any point O is the plane OPQ where P, Q are the points of contact of the planes (l, O) (m, O). Prove also that, conversely, the nul point of any plane which cuts l, m in P, Q is the pole of PQ *w. r. t* the section.

693. (R. Vythynathaswamy, B.A.) :—If ABC, A′B′C′ be triangles inscribed in the same circle, L_1, L_2, L_3 the latera-recta of parabolas having A,B,C for foci and touching the sides of A′B′C′, L_1', L_2', L_3' the latera-recta of parabolas having A′,B′,C′, as foci and touching the sides of ABC, shew that $L_1L_2L_3 = L_1'L_2'L_3'$.

694. (M. Bhimasena, Rao) :—Tangents at P and Q on a cubic curve intersect on the curve. Through Q a variable line cuts the curve in R and R′. Prove that PR and PR′ are conjugate lines of a pencil in involution.

Deduce that the four lines formed by joining any fixed point on a cubic curve to the points of contact of tangents drawn from a variable point on the curve (to itself), taken two by two, form three pencils in involution, and determine the double rays of the pencils.

695. (R. SRINIVASAN, M.A.) :—From a point P on a sphere four straight lines PA, PB, PC, PD are drawn to meet it in A, B, C, D such that $A\hat{P}B = B\hat{P}C = C\hat{P}D = D\hat{P}A$; shew that $PA + PC = PB + PD$.

696. (S. KRISHNASWAMY AIYANGAR) :—If λ, μ be the latera-recta of the parabola and the rectangular hyperbola of closest contact with a curve at any point, prove that $2\lambda\rho = \mu^2$.

697. (S. KRISHNASWAMY AIYANGAR) :—The base BC of a triangle is given in magnitude and position. If the intercept on the tangent to the incircle parallel to the base and terminated by the sides is of constant length, find the locus of the vertex.

698. (D. KRISHNAMURTI) :—A variable tetrahedron has its vertices lying on four fixed straight lines passing through a point O. If ABCD be the vertices in any position and $\alpha, \beta, \gamma, \delta$ the tetrahedral co-ordinates of a fixed point with reference to ABCD, show that

$$\frac{(OA \cdot OB \cdot OC \cdot OD)\alpha\beta\gamma\delta}{(\alpha \cdot OA + \beta \cdot OB + \gamma \cdot OC + \delta \cdot OD)^4} \text{ is constant.}$$

699. (S. RAMANUJAN) :—Show that the roots of the equations

$$\text{(i)}\quad x^6 - x^3 + x^2 + 2x - 1 = 0$$

$$\text{(ii)}\quad x^6 + x^5 - x^3 - x^2 - x + 1 = 0$$

can be expressed in terms of radicals.

700. (S. RAMANUJAN) :—Sum the series

$$(a+b+1)\left(\frac{a}{b}\right)^2 + (a+b+3)\left\{\frac{a(a+1)}{b(b+1)}\right\}^2 + (a+b+5)\left\{\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\right\}^2 + \ldots \text{to } n \text{ terms.}$$

701. (K. APPUKUTTAN ERADY, M.A.) :—P is a point within the area bounded by two confocal ellipses of semi-axes (a, b) and (a', b'). Prove that, if S, H be the common foci, the mean value of $(SP. HP)^{-1}$ is

$$\frac{2}{a'b' - ab} \cdot \log\frac{(a'+b')}{(a+b)};$$

and that of (SP.HP) is

$$\frac{1}{4(a'b'-ab)}\left\{2(a^2-b^2)\log\left(\frac{a'+b'}{a+b}\right)+a'b'(a'^2+b'^2)-ab(a^2+b^2)\right\}.$$

702. (Martyn M. Thomas) :—Prove that

$$\text{(i)} \int_0^{\frac{\pi}{2}} \sin x\,(\log \sin x).\cos x \log(\cos x)\,dx=\frac{12-\pi^2}{48}.$$

$$\text{(ii)} \int_0^{\infty} \frac{\sin x \log(\cos x)}{x}\,dx=\pi \log\frac{1}{\sqrt{2}}.$$

703. (N. Sankara Aiyar, M.A.) :—If AP is the symmedian through A, prove that

$$\Sigma\left\{(b^2+c^2)\mathrm{AK.KP}\right\}=6\,a^2b^2c^2.$$

704. (S. Malhari Rao, B.A.) :—If the sum of a number of three digits and the number formed by reversing the digits be divisible by 37, shew that the sum of all such pairs of number is 480 × 37.

705. (S. Malhari Rao, B.A.) :—The circumferences of four concentric circles are cut by two diameters. Arrange the numbers 1, 2, 3,...16 at the points of intersection in such a way that the sum of the numbers on each radius and on each circumference may be the same, and also the sum of every pair of diametrically opposite numbers may be the same.

List of Periodicals Received.

(From 16th July to 15th September 1915.)

1 American Journal of Mathematics, Vol. 37, No. 2, April 1915

2. Annals of Mathematics, Vol. 16, No. 4, June 1915.

3. Astrophysical Journal, Vol. 41, No. 4, May 1915.

4 Bulletin of the American Mathematical Society, Vol. 21, No. 10, July 1915, (2 Copies)

5. Bulletin des Sciences Mathematiques, Vol. 39, June and July 1915.

6. Educational Times, July and August 1915, (6 Copies).

7. L'Intermediaire des Mathematiciens, Vol. 22, No. 5, May 1915.

8. Journal de Mathematiques Elementaires, Vol. 39, Nos. 17, 18 and 20.

ille's Journal Vol. 1, No. 1.

ematical Gazette, Vol. 8, No. 118, July 1915, (4 Copies).

matics Teacher, Vol. 7, No. 4, June 1915.

nger of Mathematics, Vols. 44 and 45, Nos. 12, 1 and 2, April, May d June 1915.

sophical Magazine, Vol. 30, Nos. 175 and 176, July and August 15.

ar Astronomy, Vol. 23, No. 7, August September 1915, (3 Copies).

edings of the London Mathematical Society, Vol. 14, No. 4, July 15.

edings of the Royal Society of London, Vol. 91, Nos. 630, 631 and 632, June, July and August 1915.

17. Quarterly Journal of Mathematics, Vol 46, No. 3, July 1915.

18. Transactions of the American Mathematical Society, Vol. 16, No. 3, July 1915.

19. Transactions of the Royal Society of London, Vol. 215, Nos. 531 and 532.

20. Bulletin of the Calcutta Mathematical Society, Vol. 3.

21. The Tohoku Mathematical Journal, Vol. 7 and 8, Nos. 3, 4 and 1, June July 1915.

22. American Mathematical Monthly, Vol. 22, Nos. 5 and 6, May and June 1915.

The Folium of Descartes.

BY M. T. NARANIENGAR.

1. Let the equation of the curve be

$$x^3+y^3=3\,axy;$$

then the tangent at P (x_1, y_1) meets the curve at a point T (x_3, y_3) such that

$$x_3=\frac{x_1y_1^3}{x_1^3-y_1^3};\ y_3=\frac{y_1x_1^3}{y_1^3-x_1^3}.$$

If the tangent at Q (x_2, y_2) should meet the curve at the same point, we must, similarly, have

$$x_3=\frac{x_2y_2^3}{x_2^3-y_2^3};\ y_3=\frac{y_2x_2^3}{y_2^3-x_2^3}.$$

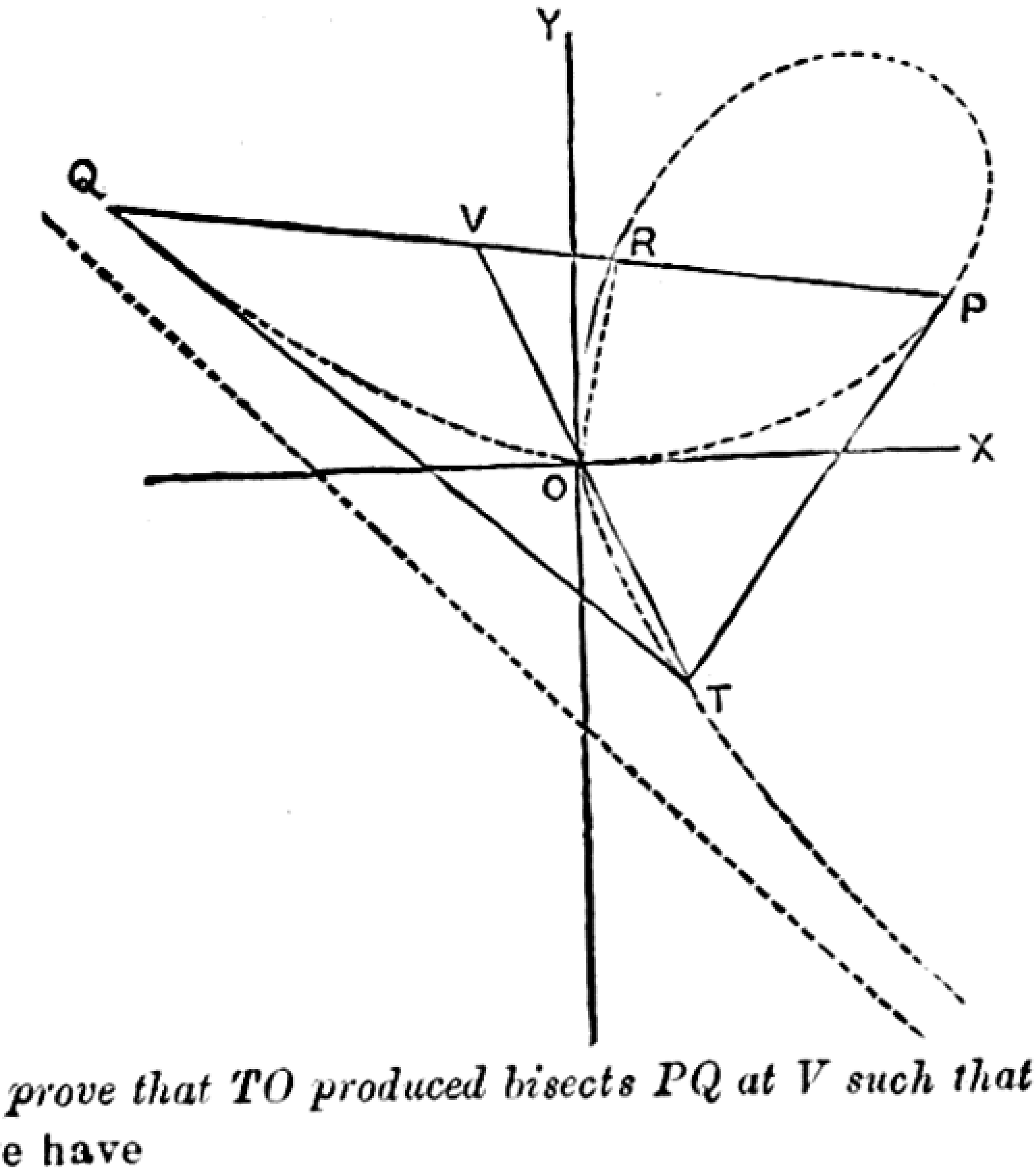
Hence, we deduce

$$\frac{y_3}{x_3}=-\frac{x_1^2}{y_1^2}=-\frac{x_2^2}{y_2^2}.$$

In other words

$$x_1:\ y_1=-x_2:\ y_2;$$

or in the figure, OP, OQ are equally inclined to the axis of x.



2. *To prove that TO produced bisects PQ at V such that OV=TO.*

For, we have

$$x_3=\frac{x_1(y_1/x_1)^3}{1-(y_1/x_1)^3}=\frac{x_1(-y_2/x_2)^3}{1+(y_2/x_2)^3}=\frac{-x_1y_2^3}{3ax_2y_2}$$
$$=-x_1+x_1x_2^2/3ay_2.$$

Similarly,

$$x_3 = -x_2 + x_2 x_1^2/3ay_1.$$

Hence, by addition,

$$2x_3 = -(x_1 + x_2.)$$

In the same way

$$2y_3 = -(y_1 + y_2).$$

Thus, the middle point of PQ is $(-x_3, -y_3)$ which proves the result.

3. *To find the equation of PQ in terms of the co-ordinates of T.*

The equation of PQ is written

$$\frac{Y+y}{X+x} = \frac{y_1-y_2}{x_1-x_2},$$

if the point T is taken as (x,y), since PQ passes through V or $(-x, -y)$.

Now $$y_1 - y_2 : x_1 + x_2 = y_1 : x_1 \text{ by § 1.}$$

$$\begin{aligned} \therefore \quad y_1 - y_2 &= -2xy_1/x_1 = -2y_1^4/(x_1^3 - y_1^3) \\ &= +2y_1y_2^3/(x_2^3 + y_2^3) \\ &= \quad 2y_1y_2^2/3ax_2. \end{aligned}$$

Similarly

$$x_1 - x_2 = 2x_1x_2^2/3ay_2.$$

$$\therefore \quad y_1 - y_2 : x_1 - x_2 = y_1y_2^3 : x_1x_2^3 = -x^2 : y^2 \text{ by § 1.}$$

Hence, the equation of PQ is

$$\frac{Y+y}{X+x} + \frac{x^2}{y^2} = 0;$$

or

$$Xx^2 + Yy^2 + 3axy = 0.$$

4. *If PQ cuts the curve in R, then OR and OT are equally inclined to the axis of x.*

This follows easily from the preceding. For, the homogeneous equation

$$xy(X^3 + Y^3) + XY(Xx^2 + Yy^2) = 0$$

should represent the three lines OP, OQ and OR.

Putting $Y = \mu X$ and $y = mx$, we have

$$m(\mu^3 + 1) + \mu(1 + m^2\mu) = 0,$$

or

$$(\mu + m)(\mu^2 m + 1) = 0,$$

which shows that OR corresponds to the value $-m$, since OP, OQ correspond to the values $\pm 1/\sqrt{-m}$ of μ. Hence the result stated.

5. *Parametric expressions for the co-ordinates of T, P, Q, R.*

Putting $y = mx$, the co-ordinates of T are easily obtained in the form

$$\left\{ \frac{3am}{m^3+1}, \frac{3am^2}{m^3+1} \right\}$$

Calling T the point 'm,' we may write the points P, Q, R as $+(\sqrt{-m})$, $(-\sqrt{-m})$ and $(-m)$ respectively.

Cor. 1. The chord of contact of the point 'm' may therefore be expressed in the form

$$X+Y\,m^2+3\,a\,m=0.$$

Cor. 2. The tangents at T and R meet the curve at the point $(-m^2)$ and the equation of TR is

$$X-Y\,m^4-3\,am^2=0.$$

Cor. 3. The envelope of PQ or TR is the rectangular hyperbola

$$4\,XY=9a^2.$$

6. *The trilinear equation of the Folium referred to OPQ as the triangle of reference.*

Denote OPQ by ABC. Then, remembering that OP, OQ and OV, OR are equally inclined to the axes which are tangents at O, we note that

(i) A is a node the tangents at which are $\beta^2-\gamma^2=0$.

(ii) If BC cuts the curve at D and E is the middle point of BC, AD is the symmedian through A or $c\beta-b\gamma=0$.

Hence, the equation of the curve is written

$$a\alpha(\beta^2-\gamma^2)+4\beta\gamma(c\beta-b\gamma)=0. \qquad \text{(Basset, § 91.)}$$

Note.—It must be borne in mind that a, b, c are not independent of one another. For, from the parametric expressions for P and Q in § 5 (where t replaces a to avoid confusion of symbols), we find

$$a=\text{PQ}=\sqrt{\left\{\left(\frac{6t\mu}{1-\mu^6}\right)^2+\left(\frac{6t\mu^5}{1-\mu^6}\right)^2\right\}}=\frac{6t\mu}{1-\mu^6}\sqrt{1+\mu^8},$$

$$b=\text{OQ}=\frac{3t\mu}{1-\mu^3}\sqrt{1+\mu^2},$$

$$c=\text{OP}=\frac{3t\mu}{1+\mu^3}\sqrt{1+\mu^2},$$

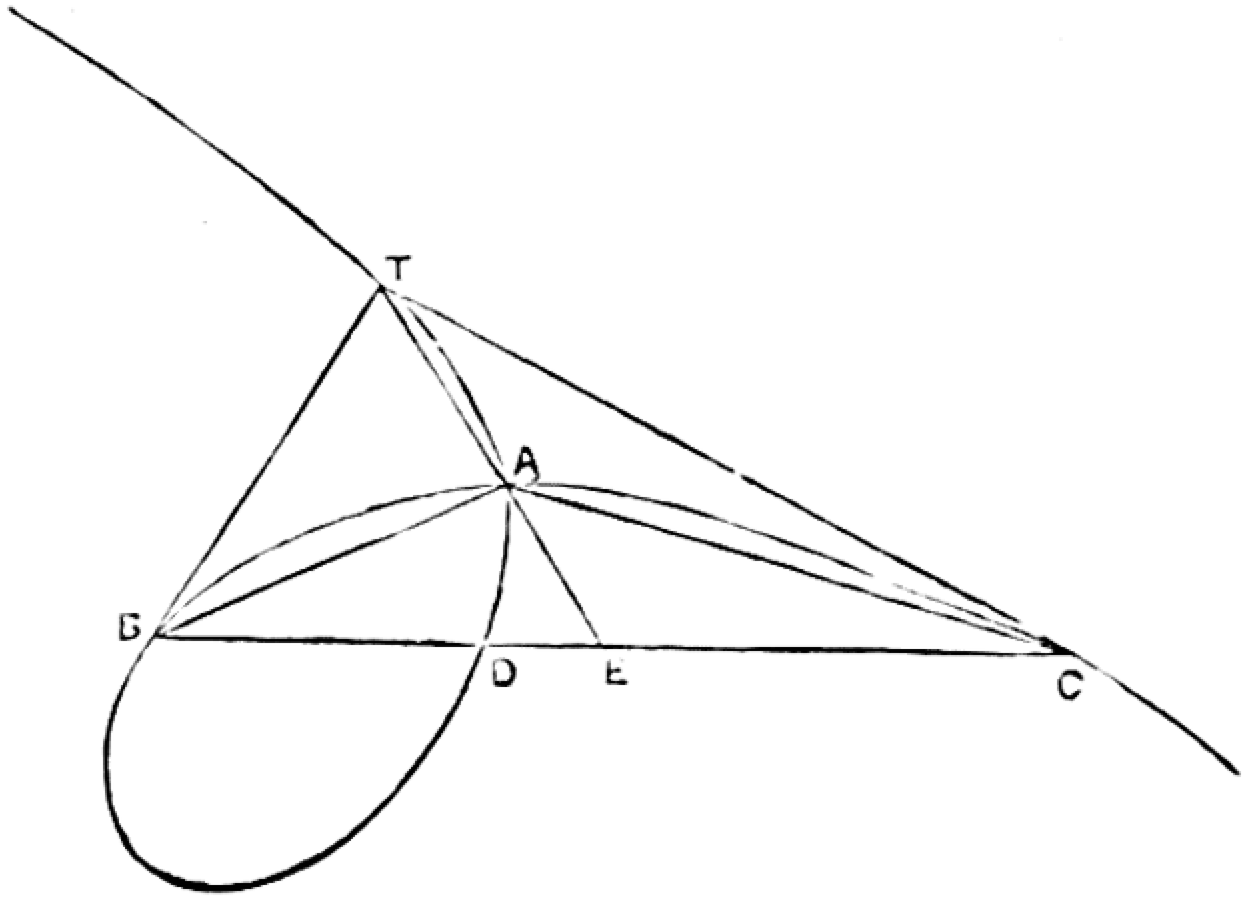
μ standing for tan POX or cot $\frac{1}{2}$ A.

Hence, the relation connecting a, b, c is found by eliminating μ from the above; thus

$$\mu^3=\left(\frac{b-c}{b+c}\right)=\cot^3\tfrac{1}{2}\text{A}=\frac{s^{\frac{3}{2}}(s-a)^{\frac{3}{2}}}{(s-b)^{\frac{3}{2}}(s-c)^{\frac{3}{2}}}.$$

The equation of the Folium is, therefore,

$$\alpha(\beta^2-\gamma^2)\sqrt{\left(\frac{1+\mu^8}{1+\mu^2}\right)}+2\beta\gamma\{\beta(1-\mu^2)-\gamma(1+\mu^2)\}=0.$$



7. *If any chord pDq be drawn through D, then the isogonal conjugates of Ap, Aq, are harmonically conjugate to Aq, Ap respectively.*

For, the equation of pDq is of the form

$$c\beta-b\gamma=k\alpha,$$

and where this cuts the cubic

$$a(\beta^2-\gamma^2)+4k\beta\gamma=0.$$

That is, the lines Ap, Aq are of the form

$$\beta=\lambda\gamma,\ \beta=-\gamma/\lambda;$$

whence the result stated follows.

8. *If AB′, AC′ be two isogonal chords, and B′C′ meet BC in F and the curve in D′, than A(FB′DC′)* and *A(FBD′C) are harmonic.*

Let the equation to B′C′ be $\alpha=m\beta+n\gamma$ so that AF is $m\beta+n\gamma=0$. Eliminating α between this and the equation of the curve, we get

$$a(m\beta+n\gamma)(\beta^2-\gamma^2)+4\beta\gamma(c\beta-b\gamma)=0$$

for the lines AB′, AD′, AC′.

Now, if this cubic has a factor $m\beta-n\gamma$, the remaining quadratic factor should be of the form

$$(\beta^2+\gamma^2+\lambda\beta\gamma)$$

as is seen by division. In other words, if AD′ be the harmonic conjugate of AF, the equation of AB′, AC′ is

$$\beta^2+\gamma^2+\lambda\beta\gamma=0,$$

that is AB′, AC′ are isogonally conjugate. This proves the first result.

Again, since the tangents at B′, C′ meet on the curve by § 1, we may consider AB′C′ as the triangle of reference and AB, AC as a pair of isogonals. Hence from the preceding it follows that the pencil A(FBD′C) is also harmonic.

9. *If through B or C a chord pq be drawn parallel to the median AE, then Ap, Aq are isogonal.*

For, the equation to the chord through B parallel to AE is

$$a\alpha+b\beta+c\gamma-(b\beta-c\gamma)=0,$$

or

$$a\alpha+2c\gamma=0;$$

and this cuts the curve in points lying on

$$-2c\gamma(\beta^2-\gamma^2)+4\beta\gamma(c\beta-b\gamma)=0$$

i.e.

$$\gamma(c\beta^2+c\gamma^2-2b\beta\gamma)=0.$$

Hence the equation to Ap, Aq is

$$c(\beta^2+\gamma^2)-2b\beta\gamma=0,$$

which shows that they are isogonal to each other.

10. The equation of the curve is written

$$\beta^2(a\alpha+4c\gamma)=\gamma^2(a\alpha+4b\beta),$$

i.e.

$$\beta^2.v=\gamma^2.w,$$

where v, w donote the tangents at B, C respectively.

Hence: *The curve is the locus of points such that the ratio of the perpendiculars on TB, TC varies as the duplicate ratio of those on AB, AC.*

11. *If any chord Tpq be drawn through T then Ap, Aq are harmonically conjugate* to *AB, AC.*

Any line through T is $v=k^2w$ say.

Putting $v=k^2w$ in the equation $\beta^2v=\gamma^2w$, we have

$$k^2\beta^2=\gamma^2, \text{ or } k\beta=\pm\gamma,$$

for the lines Ap, Aq. Hence the result stated.

12. *Other Transformations* :

The equation $x^3+y^3=3\ axy$ is transformed into

$$x(x^2+3y^2)=a(x^2-y^2)$$

by turning the axes through 45°.

Putting $x=\rho\cosh u$, $y=\rho\sinh u$ in the latter, we get

$$\rho\cosh 3\,u=a,$$

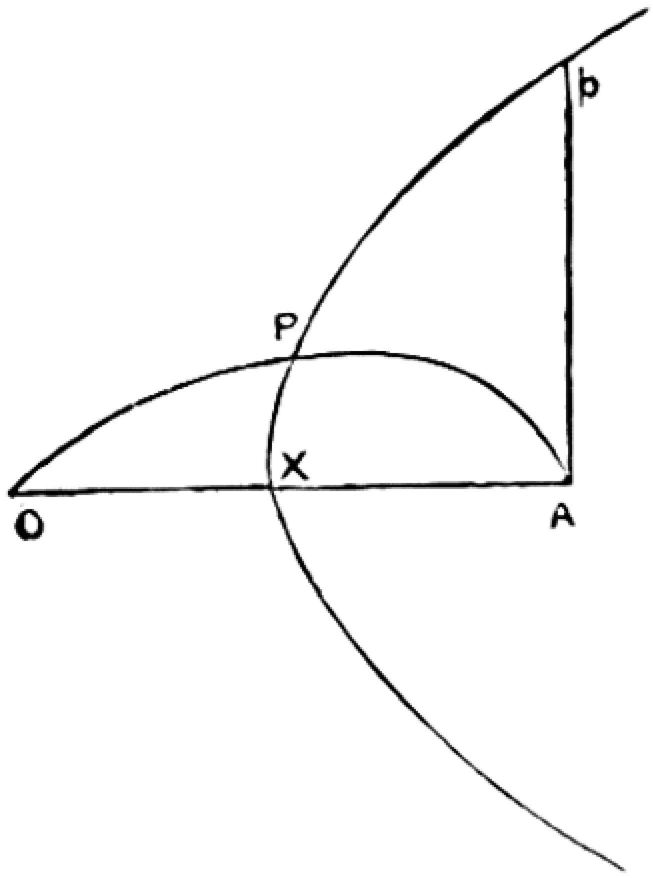
since $x^2-y^2=\rho^2$.

13. *Geometrical Interpretation of the equation $\rho\cosh 3u=a$.*

Let the rectangular hyperbola

$$x^2-y^2=\rho^2$$

be drawn, where $OX=\rho$. Then, if $OA=a=\rho\cosh 3u$, the curvilinear area $OXp=\frac{1}{2}\rho^2.\,3u$; so that P or $(\rho\cosh u,\ \rho\sinh u)$ is such that area $OXP=\frac{1}{3}$ area OXp.



In other words, the point P of trisection of the curvilinear area OXp traces the Folium

$$\rho\cosh 3u=a. \quad \ldots \quad \ldots \quad (1)$$

14. The tangent at 'u_1' may be written

$$a/\rho=2\cosh(u+2u_1)-\cosh(u-4u_1) \ldots \quad \ldots \quad (2)$$

This meets (1) at points given by

$$\cosh 3u=2\cosh(u+2u_1)-\cosh(u-4u_1)$$

i.e.

$$\cosh 3u+\cosh(u-4u_1)=2\cosh(u+2u_1)$$

or

$$2\cosh 2(u-u_1).\cosh(u+2u_1)-2\cosh(u+2u_1)=0.$$

$$\therefore \quad \cosh(u+2u_1)=0.$$

$$\therefore \quad u+2u_1=i\,(2n+1)\pi/2. \quad \ldots \quad \ldots \quad (3)$$

If the tangent at 'u_2' should meet the curve at the same point as determined by (3)

$$u+2u_2=i(2n'+1)\pi/2$$

$$\therefore \quad u_1-u_2=i(n'-n)\pi/2$$

or

$$\cosh(u_1-u_2)=0 \text{ or } \pm 1,$$

which agrees with the result of § 1.

Again, the tangent from a point $(\rho_0,\ u_0)$ is such that

$$a/\rho_0=2\cosh(u_0+2u)-\cosh(u_0-4u) \quad \ldots \quad \ldots \quad (4)$$

u being the point of contact.

Hence, the four points of contact u_1, u_2, u_3, u_4 satisfy equation (4), and are such that

$$\Sigma(\tanh u_1)=0,$$

which is identical with the property that the sum of the tangents of vectorial angles of the four points of contact is zero.

In particular if (ρ_o, u_o) be on the curve

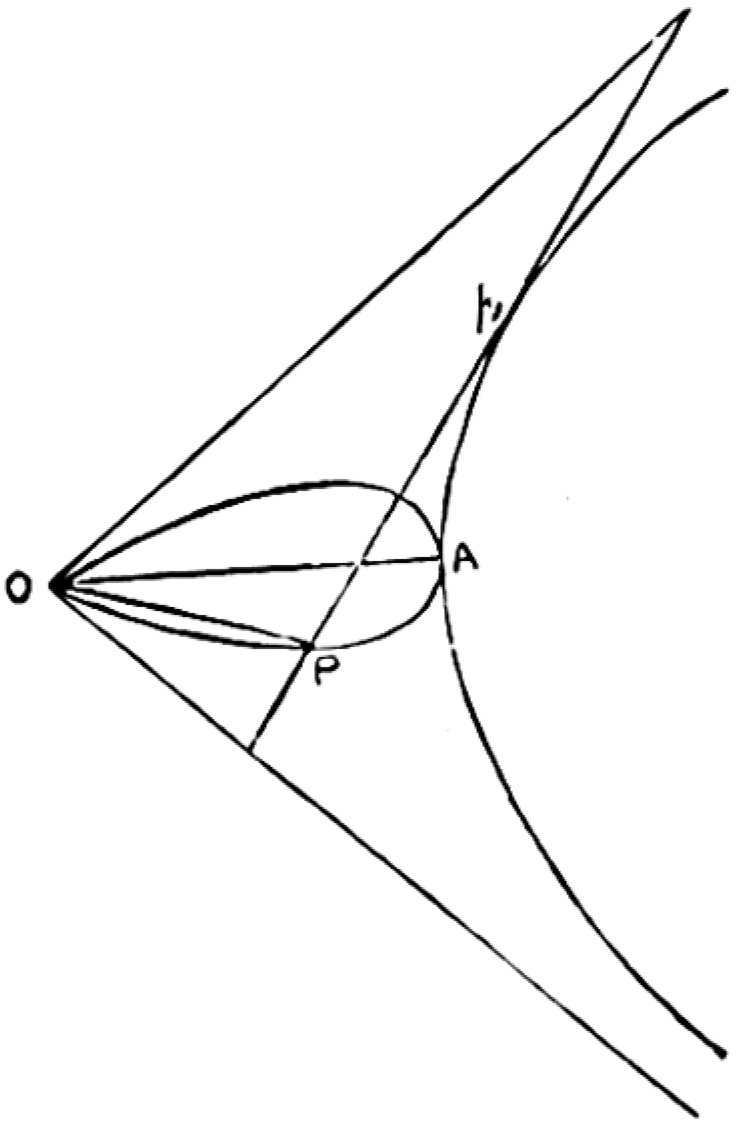
$$2 \tanh u_o + \tanh u_1 + \tanh u_2 = 0.$$

15. The line $\rho \cosh (u-u_1)=a$, which is a tangent to the hyperbola $\rho=a$, meets the curve in point given by

$$\cosh 3u = \cosh (u-u_1)$$

whence $u=-\frac{1}{2}u_1$.

In cartesians this means that the tangent at $(\sec\alpha, \tan\alpha)$ to the hyperbola $x^2-y^2=a^2$ cuts the Folium at the point whose vectorial angle is $(-\frac{1}{2}\alpha)$.



Hence the following simple construction for the Folium :—

Take the tangent at the point p $(\sec\alpha, \tan\alpha)$; make $AOP=-\frac{1}{2}\alpha$. Then the locus of P is the Folium

$$\rho \cosh 3u = a.$$

SHORT NOTES.

On the product $\prod_{n=0}^{n=\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$.

I. Let

$$\phi(\alpha,\beta)=\left\{1+\left(\frac{\alpha+\beta}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{2+\alpha}\right)^3\right\} \qquad \dots \quad (1)$$

It is easy to see that

$$\left\{1+\left(\frac{\alpha+\beta}{n+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{n+\beta}\right)^3\right\}=\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\times$$

$$\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]=$$

$$\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]; \qquad \dots \quad (2)$$

$$\prod_{n=1}^{n=\infty}\left\{\frac{\left(1+\frac{\alpha+2\beta}{n}\right)\left(1+\frac{\beta+2\alpha}{n}\right)}{\left(1+\frac{\alpha}{n}\right)^3\left(1+\frac{\beta}{n}\right)^3}\right\}=$$

$$\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}; \qquad \dots \quad (3)$$

and

$$\prod_{n=1}^{n=\infty}\left[1-\left\{\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]$$

$$\left[1-\left\{\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2n}\right\}^2\right]=$$

$$\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}. \qquad \dots \quad (4)$$

It follows from (1)–(4) that

$$\phi(\alpha,\beta)\,\phi(\beta,\alpha)=\frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}\times$$

$$\left\{\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}\right\}. \qquad \dots \quad (5)$$

But it is evident that, if $\alpha-\beta$ be any integer, then $\phi(\alpha,\beta)/\phi(\beta,\alpha)$ can be expressed in finite terms. From this and (5) it follows that

$\phi(\alpha, \beta)$ can be expressed in finite terms, if $\alpha-\beta$ be any integer. That is to say

$$\left\{1+\left(\frac{x}{a}\right)^3\right\}\left\{1+\left(\frac{x}{a+d}\right)^3\right\}\left\{1+\left(\frac{x}{a+2d}\right)^3\right\}\cdots\cdots\cdots$$

can be expressed in finite terms if $x-2a$ be a multiple of d.

2. Suppose now that $\alpha=\beta$ in (5). We obtain

$$\left\{1+\left(\frac{2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{3+\alpha}\right)^3\right\}\cdots$$
$$=\frac{\{\Gamma(1+\alpha)\}}{\Gamma(1+3\alpha)}\cdot\frac{\sinh \pi\alpha\sqrt{3}}{\pi\alpha\sqrt{3}}. \quad (6)$$

Similarly putting $\beta=\alpha+1$ in (5), we obtain

$$\left\{1+\left(\frac{2\alpha+1}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha+1}{2+\alpha}\right)^3\right\}\cdots\cdots\cdots\cdots$$
$$=\frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(2+3\alpha)}\cdot\frac{\cosh \pi(\frac{1}{2}+\alpha)\sqrt{3}}{\pi}. \quad \ldots (7)$$

Again, since

$$\left\{1+\left(\frac{\alpha}{n}\right)^3\right\}\left\{1+3\left(\frac{\alpha}{2n+\alpha}\right)^2\right\}=\frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\alpha^2}{n^2}+\frac{\alpha^4}{n^4}\right)}{\left(1+\frac{\alpha}{2n}\right)^2},$$

it is easy to see that

$$\left[\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\cdots\right]\left[\left\{1+3\left(\frac{\alpha}{2+\alpha}\right)^2\right\}\left\{1+3\left(\frac{\alpha}{4+\alpha}\right)^2\right\}\cdots\right]$$
$$=\frac{\Gamma(\frac{1}{2}\alpha)}{\Gamma\{\frac{1}{2}(1+\alpha)\}}\left(\frac{\cosh \pi\alpha\sqrt{3}-\cos \pi\alpha}{2^{\alpha+2}\pi\alpha\sqrt{\pi}}\right). \quad (8)$$

3. It is known that, if the real part of α is positive, then

$$\log\Gamma(\alpha)=(\alpha-\tfrac{1}{2})\log\alpha-\alpha+\tfrac{1}{2}\log 2\pi+2\int_0^\infty\frac{\tan^{-1}(x/\alpha)}{c^{2\pi x}-1}dx. \quad (9)$$

From this we can show that, if the real part of α is positive, then

$$\tfrac{1}{2}\log 2\pi\alpha+\frac{\pi\alpha}{\sqrt{3}}+\log\left\{\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\left(1+\frac{\alpha^3}{3^3}\right)\cdots\right\}$$
$$=\log\left(\frac{\cosh \pi\alpha\sqrt{3}-\cos \pi\alpha}{\pi\alpha}\right)+2\int_0^\infty\frac{\tan^{-1}(x/\alpha)^3}{e^{2\pi x}-1}dx. \quad (10)$$

From this and the previous section it follows that

$$\int_0^\infty\frac{\tan^{-1}x^3}{e^{2\pi n x}-1}dx$$

can be expressed in finite terms if n is a positive integer. Thus for example,

$$\int_0^\infty \frac{\tan^{-1}x^3}{e^{2\pi x}-1}dx=\tfrac{1}{4}\log 2\pi-\frac{\pi}{4\sqrt{3}}-\tfrac{1}{2}\log\left(1+e^{-\pi\sqrt{3}}\right); \qquad (11)$$

$$\int_0^\infty \frac{\tan^{-1}x^3}{e^{4\pi x}-1}dx=\tfrac{1}{8}\log 12\pi-\frac{\pi}{4\sqrt{3}}-\tfrac{1}{4}\log\left(1-e^{-2\pi\sqrt{3}}\right); \qquad (12)$$

and so on

4. It is also easy to see that

$$\frac{1^2}{1^3+n^3}-\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}-\frac{4^2}{4^3+n^3}+\dots\dots\dots$$

$$=\frac{1}{3}\left(\frac{1}{1+n}-\frac{1}{2+n}+\frac{1}{3+n}-\frac{1}{4+n}+\dots\dots\right)$$

$$+\frac{4}{3}\left\{\frac{2-n}{(2-n)^2+3n^2}-\frac{4-n}{(4-n)^2+3n^2}+\frac{6-n}{(6-n)^2+3n^2}-\dots\right\}\dots \qquad (13)$$

Since

$$\frac{\pi}{4\cosh\frac{1}{2}\pi x}=\frac{1}{1^2+x^2}-\frac{3}{3^2+x^2}+\frac{5}{5^2+x^2}-\dots\dots$$

it is clear that the left hand side of (13) can be expressed in finite terms if n is any odd integer. For example

$$\frac{1^2}{1^3+1}-\frac{2^2}{2^3+1}+\frac{3^2}{3^3+1}-\frac{4^2}{4^3+1}+\dots=\frac{1}{3}\left(1-\log 2+\pi\operatorname{sech}\tfrac{1}{2}\pi\sqrt{3}\right). \qquad (14)$$

The corresponding integral in this case is

$$\int_0^\infty \frac{x^5}{\sinh\pi x}\cdot\frac{dx}{n^6+x^6}=\frac{2}{\pi}\int_0^\infty\left\{\frac{1}{2x^2}+\sum_{\nu=1}^{\nu=\infty}\frac{(-1)^\nu}{\nu^2+x^2}\right\}\frac{x^6dx}{n^6+x^6}$$

$$=\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}+\dots\dots\right)$$

$$-\frac{4}{3}\left\{\frac{n+2}{(n+2)^2+3n^2}-\frac{n+4}{(n+4)^2+3n^2}+\frac{n+6}{(n+6)^2+3n^2}-\dots\right\}; \qquad (15)$$

and so the integral on the left hand side of (15) can be expressed in finite terms if n is any odd integer. For example,

$$\int_0^\infty \frac{x^5}{\sinh\pi x}\cdot\frac{dx}{1+x^6}=\frac{1}{3}\left(\log 2-1+\pi\operatorname{sech}\tfrac{1}{2}\pi\sqrt{3}\right). \qquad \dots (16)$$

S. Ramanujan.

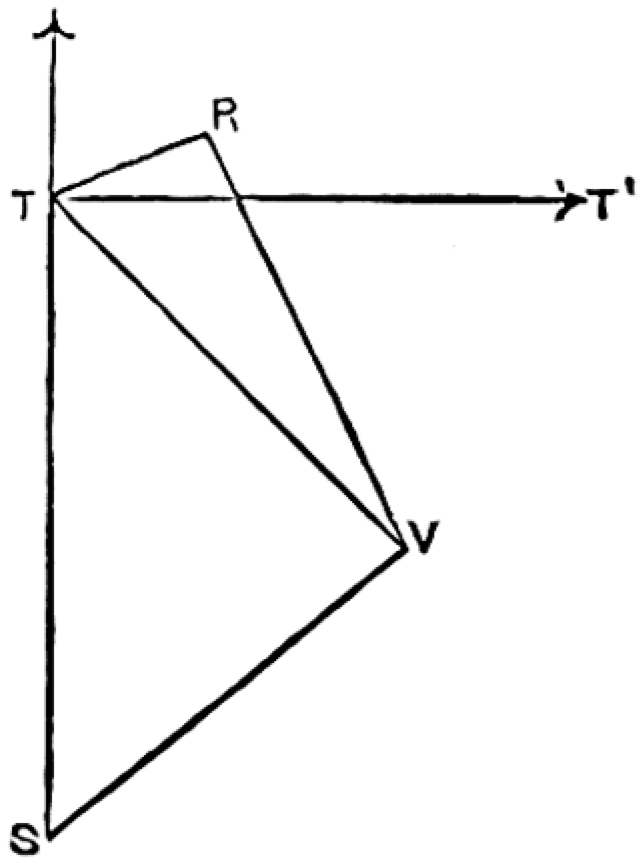
Planetary Aberration.

The following method of obtaining the expression for planetary aberration is somewhat simpler than the one given by Ball, and is from elementary considerations.

Let S be the sun, T the earth moving in the direction TT′ perpendicular to ST with velocity v. The planet V is moving in the direction VV′ tangential to its orbit with velocity v'. If $ST=r$, $SV=r'$, then $v\sqrt{r}=v'\sqrt{r'}$.

Let $S\hat{T}V=E$, $S\hat{V}T=P$.

Impress on the earth a velocity equal and opposite to that of V.



Then the velocities of the earth along TT′ and ST are

$$v+v'\cos(E+P),\ v'\sin(E+P)$$

Let TR represent the direction and magnitude of the relative velocity of the earth, and VR the velocity of light on the same scale.

Then the planetary aberration $\epsilon=T\hat{V}R$.

Now, $\dfrac{\sin\epsilon}{TR}=\dfrac{\sin(90°-E+\theta)}{\mu}$, where $\theta=R\hat{T}T'$.

$$\begin{aligned}\therefore\quad \mu\sin\epsilon &= TR\cos(E-\theta)\\ &= \{v+v'\cos(E+P)\}\cos E+v'\sin(E+P)\sin E\\ &= v\cos E+v'\cos P.\end{aligned}$$

If v_o denote the velocity of a planet at distance r_o from the sun, we have

$$\epsilon=\frac{v_o\sqrt{r_o}}{\mu}\left\{\frac{\cos E}{\sqrt{r}}+\frac{\cos P}{\sqrt{r'}}\right\}.$$

P. R. Krishnaswami.

Note on Planetary Aberration.

[The following note was suggested by Professor Turner's '*Notes on Aberration*' in the *Monthly Notices* of the Royal Astronomical Society, April 1909.]

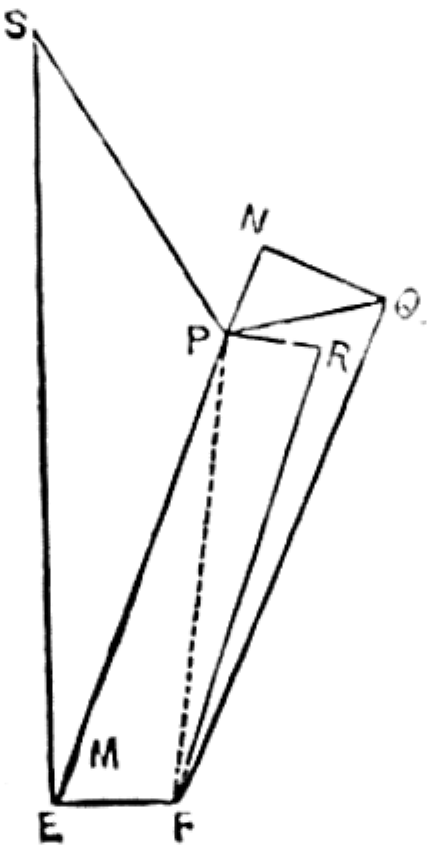
1. Let S be the sun and P, E simultaneous positions of a planet and the earth; and let a ray of light which leaves the planet in the position P reach the earth when it goes to F. Also, let Q be the position of the planet when the earth is at F; so that the angles SEF, SPQ are nearly right angles. Then within the time taken by the earth to go from E to F, the planet goes from P to Q and the planetary ray travels from P to F. That is, if t be this time

$$EF = et,\ PQ = vt,\ PF = V't, \quad \ldots \quad \ldots \ (1)$$

where e, v, V' are the velocities of the earth, the planet and the planetary ray respectively.

Now, the velocity V' of the planetary ray is that due to the actual velocity V of light combined with the velocity v of the planet. Hence, we have

$$QF = Vt. \quad \ldots \quad \ldots \ (2)$$



Complete the parallelogram PEFR. Then the ray which is actually travelling along PF with velocity V' appears to an observer on the earth to be proceeding in the direction RF.

In other words, the planet's *apparent* direction FR, is while its *true* direction is FQ; and the angle α between these directions is the value of the aberration of the planet at the instant of observation and is identical with the angle between the *true* positions of the planet at E and F—what amounts to the same thing, its *apparent* position at F and *apparent* position t seconds later.

2. The value of t is found in terms of the actual distance of the planet from the earth at the instant of observation *viz.*, QF by the formula (2), and we may write

$$t = (a^2 + b^2 - 2ab \cos \Theta)^{\frac{1}{2}}/V \quad \ldots \quad \ldots \ (3)$$

where a, b, Θ denote SF, SQ, angle FSQ respectively.

3. *To find a formula for* α.

With the usual convention of signs α is positive or negative according as it is measured contraclockwise or clockwise. Thus in the figure α should be regarded as negative. Drop the perpendiculars QN, FM on EP, then we have

$$
\begin{aligned}
-FQ \sin \alpha &= QN - FM \\
&= PQ \sin NPQ - EF \sin MEF \\
&= PQ \cos SPN - EF \cos SEP.
\end{aligned}
$$

i.e. $$Vt \sin \alpha = vt \cos SPE + et \cos SEP, \text{ by (1) and (2).}$$

Hence $$V \sin \alpha = v. \cos SPE + e. \cos SEP.$$

[*Vide* : Ball's *Spherical Astron.*, § 90.]

M. T. Naraniengar.

The Face of the Sky for January and February.

Phases of the Moon.

	January. D.	H.	M.		*February.* D.	H.	M.	
New Moon	4	11	15	P.M.	3	10	36	A.M.
First Quarter ...	11	10	8	P.M.	10	4	56	P.M.
Full Moon ...	20	2	59	A.M.	18	8	59	P.M.
Last Quarter ...	27	7	5	P.M.	26	3	54	A.M.

Planets.

Mercury is an evening star in January. It is in inferior conjunction early in February and thereafter a morning star for the rest of the month.

Venus continues to be an evening star in these months. It is near the boundary between Capricornus and Aquarius in January and in Pisces in February.

Mars is in Leo in these months and is in opposition in the middle of February.

Jupiter is in Pisces in these months and is approaching the Sun. Saturn is still in Taurus in these months and is also approaching the Sun.

V. Ramesam.

SOLUTIONS.

Question 451.

(K. J. SANJANA, M.A.) :—Shew how to find pairs of isosceles triangles of rational areas such that the perimeters are in one given ratio and the areas in another given ratio.

Example : the perimeters are as 25:9 and the areas are equal.

Remarks by the Proposer.

We easily obtain the following among other solutions :—

193 (*bis*), 190, and 401 (*bis*), 798 ;
9·15834061 (*bis*), 9·19965000, and
25·12937805 (*bis*), 25·25757512 ;
9·15429061 (*bis*), 9·17535000, and
25·12127805 (*bis*), 25·24137512.

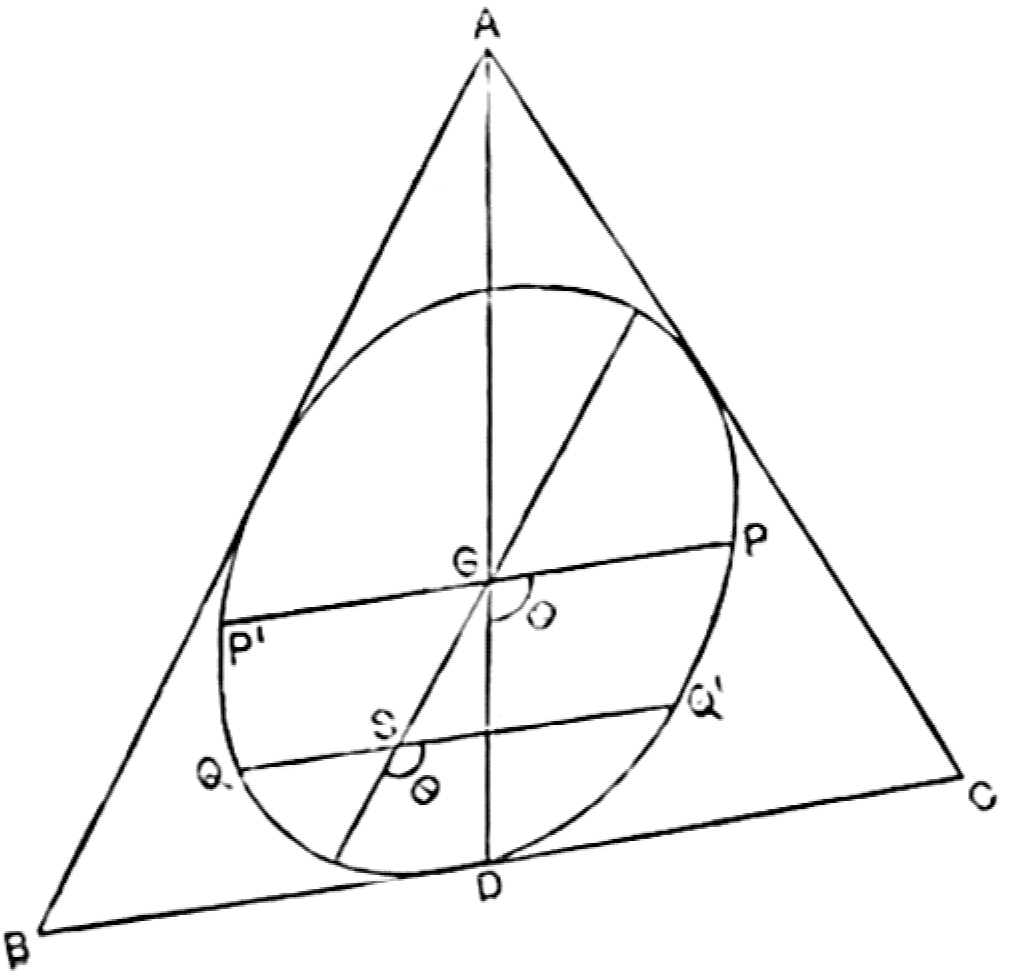
[The last pair of triangles is given by Mahaviráchárya's rule ; see ṣloka 137, Ch. VII, Rao Bahadur Rangáchárya's edition of *Gaṇita-sára-sangraha*. This rule generally gives extremely large values for the sides when expressed in integers.]

Question 539.

(S. KRISHNASWAMI AIYANGAR) :—The focal chords of the maximum inscribed ellipse of the triangle of reference parallel to the sides are the roots of the equation

$$x^3-2x^2\Delta^{\frac{1}{2}}\tan^{\frac{1}{2}}\lambda\cot w+x\Delta\tan\lambda\,\mathrm{cosec}^2 w-2R^2\Delta^{\frac{1}{2}}\tan^{\frac{3}{2}}\lambda=0$$

where λ is the minor semi—steiner angle, w is the Brocard angle and R the circumradius.



Solution by N. Sankara Aiyar, M.A.

Taking the equation of the conic in polar co-ordinates as usual, we get that the length of a focal chord is

$$\frac{l}{1-e\cos\theta}+\frac{l}{1+e\cos\theta}=\frac{2l}{1-e^2\cos^2\theta}.$$

Now, the semi-diameter of the conic parallel to this chord is given by

$$\frac{1}{r^2}=\frac{\cos^2\theta}{\alpha^2}+\frac{\sin^2\theta}{\beta^2}.$$

$$\therefore\ QQ'=\frac{2\beta^2}{\alpha(1-e^2\cos^2\theta)}=\frac{2\beta^2\alpha}{\alpha^2\sin^2\theta+\beta^2\cos^2\theta}$$

$$=\frac{2\beta^2\alpha}{\alpha^2\beta^2/r^2}=\frac{2r^2}{\alpha}=\frac{20D^2}{\alpha}$$

$$=\frac{2a^2}{12}+\frac{3}{\Delta^{\frac{1}{2}}\cot\frac{1}{2}l}.\text{(J.I.M.S., Vol. V, p. 112)}$$

$$=\frac{a^2}{2\Delta^{\frac{1}{2}}\cot\frac{1}{2}\lambda}$$

$$\therefore\ x_1=\frac{a^2}{2}\Delta^{-\frac{1}{2}}\tan^{\frac{1}{2}}\lambda.$$

$$\therefore\ x_1+x_2+x_3=\frac{a^2+b^2+c^2}{2}.\Delta^{-\frac{1}{2}}\tan^{\frac{1}{2}}\lambda.$$

$$=2\Delta\cot w\,\Delta^{-\frac{1}{2}}\tan^{\frac{1}{2}}\lambda$$

$$=2\Delta^{\frac{1}{2}}\cot w\tan^{\frac{1}{2}}\lambda.$$

$$\Sigma x_2x_3=\sum\frac{b^2c^2\Delta^{-1}}{4}\tan\lambda$$

$$=\Sigma\Delta\operatorname{cosec}^2A\tan\lambda$$

$$=\Delta\tan\lambda\operatorname{cosec}^2 w$$

$$x_1x_2x_3=\frac{a^2b^2c^2\Delta^{-\frac{3}{2}}\tan^{\frac{3}{2}}\lambda.}{8}$$

$$=2R^2\Delta^2.\Delta^{-\frac{3}{2}}\tan^{\frac{3}{2}}\lambda$$

$$=2R^2\Delta^{\frac{1}{2}}\tan^{\frac{3}{2}}\lambda.$$

Hence the required equation is

$$x^3-2x^2\Delta^{\frac{1}{2}}\tan^{\frac{1}{2}}\lambda\cot w+x\Delta\tan\lambda\operatorname{cosec}^2 w-2R^2\Delta^{\frac{1}{2}}\tan^{\frac{3}{2}}\lambda=0.$$

Question 577.

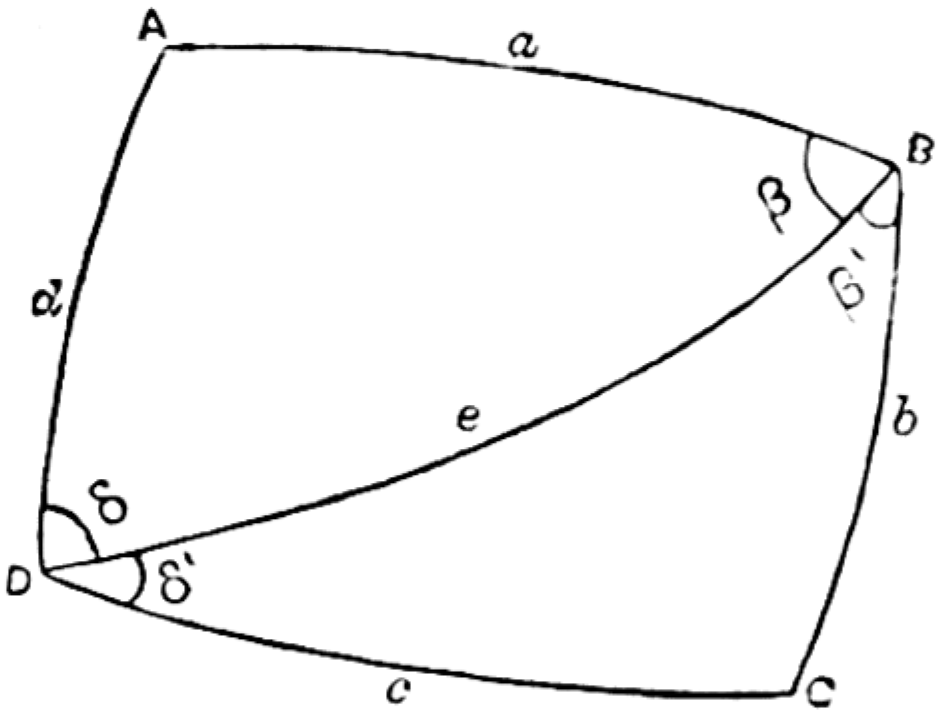
(K. Appukuttan Erady, M.A.) :—If S, S′ are the semi-sums of opposite angles of a spherical quadrilateral, prove that

$$\cos(S-S')\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}\sin\frac{d}{2}-\cos(S+S')\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\cos\frac{d}{2}$$

$$=\tfrac{1}{4}\Sigma\cos a,$$

a, *b*, *c*, *d* being the sides of the quadrilateral.

Solution by J. C. Swaminarayan, M.A.



From the above figure, we can easily see that

$$\left.\begin{aligned}
\sin\frac{A+\beta+\delta}{2} &= \frac{\cos a+\cos d+1+\cos e}{4\cos\frac{a}{2}\cos\frac{d}{2}\cos\frac{e}{2}}\\
\sin\frac{C+\beta'+\delta'}{2} &= \frac{\cos b+\cos c+1+\cos e}{4\cos\frac{b}{2}\cos\frac{c}{2}\cos\frac{e}{2}}\\
\cos\frac{A+\beta+\delta}{2} &= -\frac{\sin\frac{a}{2}\sin\frac{d}{2}\sin A}{\cos\frac{e}{2}}\\
\cos\frac{C+\beta'+\delta'}{2} &= -\frac{\sin\frac{b}{2}\sin\frac{c}{2}\sin C}{\cos\frac{e}{2}}
\end{aligned}\right\}\ \ldots\qquad\ldots\quad (1)$$

and

$$\left.\begin{aligned}
\sin\frac{A-\beta-\delta}{2} &= \frac{\cos a+\cos d-1-\cos e}{4\cos\frac{e}{2}\sin\frac{a}{2}\sin\frac{d}{2}}\\
\sin\frac{C-\beta'-\delta'}{2} &= \frac{\cos b+\cos c-1-\cos e}{4\cos\frac{e}{2}\sin\frac{b}{2}\sin\frac{c}{2}}\\
\cos\frac{A-\beta-\delta}{2} &= \frac{\cos\frac{a}{2}\cos\frac{d}{2}\sin A}{\cos\frac{e}{2}}\\
\cos\frac{C-\beta'-\delta'}{2} &= \frac{\cos\frac{b}{2}\cos\frac{c}{2}\sin C}{\cos\frac{e}{2}}.
\end{aligned}\right\}\ \ldots\qquad\ldots\quad (2)$$

Hence

$$16 \cos (S-S') \prod \sin \frac{a}{2} = \frac{\sin A \sin C \prod \sin a}{\cos^2 \frac{e}{2}}$$

$$- \frac{\{\cos b + \cos c - 1 - \cos e\} \{\cos a + \cos d - 1 - \cos e\}}{\cos^2 \frac{e}{2}}$$

and

$$16 \cos (S+S') \prod \pi \cos \frac{a}{2} = \frac{\sin A \sin C \prod \sin a}{\cos^2 \frac{e}{2}}$$

$$- \frac{\{\cos b + \cos c + 1 + \cos e\} \{\cos a + \cos d + 1 + \cos e\}}{\cos^2 \frac{e}{2}}$$

$$\therefore \cos (S-S') \prod \sin \frac{a}{2} - \cos (S+S') \prod \mathrm{oos} \frac{a}{2}$$

$$= \frac{1}{16} \frac{2(1+\cos e)(\cos a + \cos b + \cos c + \cos d)}{\cos^2 \frac{e}{2}}$$

$$= \frac{1}{4} \Sigma \cos a.$$

Question 588.

(T. P. Trivedi, M.A., LL.B.) :—If S_n denote the series $1 - \frac{1}{3^3} + \frac{1}{5^3} +$ $+ \ldots$ to n terms, prove that

$$\sum_1^\infty \frac{(-1)^{n-1} S_n}{(2n+1)^3} = \frac{\pi^6}{15 \cdot 2^{11}}.$$

Solution by A. Narasinga Rao, B.A.

Consider the identity

$$\sum_1^\infty \frac{S_n}{(2n+1)^3} = \sum \frac{1}{(2n+1)^3} \cdot \frac{(-1)^{n+r-1}}{(2r+1)^3}$$

the summation extending to all $+ve$ integral values of n and r.

Now $\sec z = \frac{2^2}{\pi^1} \Sigma_1 + \frac{2^4}{\pi^3} z^2 \Sigma_3 + \ldots + \frac{2^{2n+2}}{\pi^{2n+1}} z^{2n} \Sigma_{9n+1} + \ldots$ (i)

where $\Sigma_{2n+1} = \frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \ldots$ (Hobson's *Trig.*, p. 339.)

Comparing the co-effs. of z^2 on both sides of (i), we have $\Sigma_3 = \frac{\pi^3}{2^5}$.

But $$(\Sigma_3)^2=\left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}\cdots\right)+2\sum\frac{(-)^{n+r}}{(2n+1)^3(2r+1)^3}.$$

$$\sum\frac{S_n}{(2n+1)^3}=-\frac{1}{2}\left\{\frac{\pi^6}{2^{10}}-\frac{\pi^6}{945}\cdot\frac{63}{64}\right\}=\frac{\pi^6}{15\cdot 2^{11}}.$$

Comparing the general terms on both sides of (i), we have

$$\Sigma_{2n+1}=\frac{\pi^{2n+1}}{2^{2n+2}}\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2!} & 0 & 1 & \cdots & 0 & 0 \\ 0 & -\frac{1}{2!} & 0 & \cdots & 0 & 0 \\ \frac{1}{4!} & 0 & -\frac{1}{2!} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \\ \cdots & \cdots & \cdots & & \cdots & \\ \cdots & \cdots & \cdots & & 0 & 1 \\ \frac{(-)^n}{(2n)!} & 0 & \frac{(-)^{n-1}}{(2n-2)!} & \cdots & -\frac{1}{2!} & 0 \end{vmatrix}$$

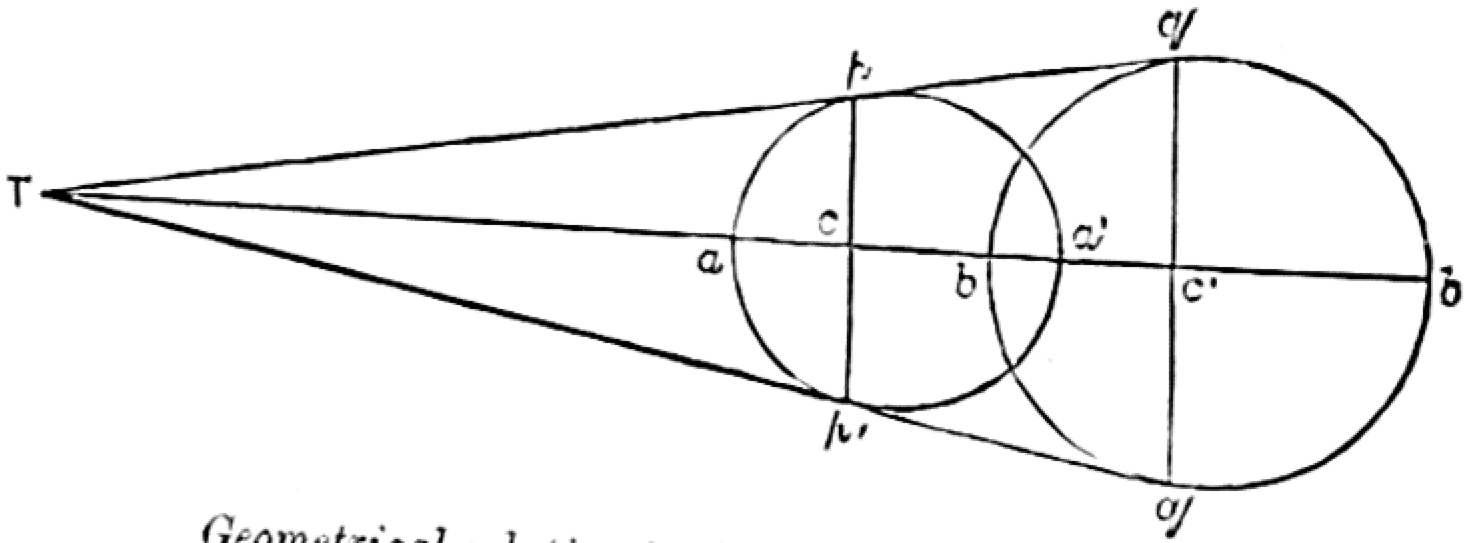
[J. I. M .S., Vol. VII, p, 52.]

We may therefore find by the same method the values of

$$\sum\frac{(-)^{m+n}}{(2m+1)^{2r+1}}\ \frac{1}{(2n+1)^{2s+1}}=(\Sigma_{2r+1})(\Sigma_{2s+1})-\sum\frac{1}{(2m+1)^{2(r+s+1)}}$$

Question 599.

(V. V. S. NARAYAN) :—If two circles cut orthogonally find the focus of a point such that the tangents from it to the two circles form a harmonic pencil.



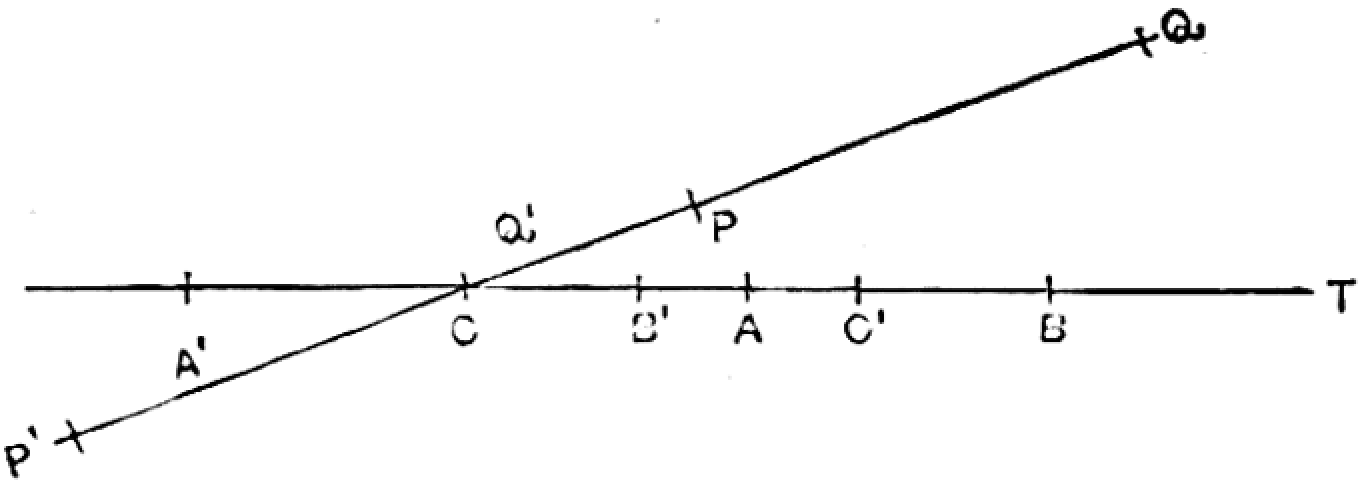
Geometrical solution by N.B. Pendse, M.A., LL.B.

Draw the common tangents to the two circles. Let them meet at T. Let $Taba'b'$ be the line of centres and let pcp' and $qc'q'$ be the chords of contact of the circles.

Reciprocate the curve with respect to the point T.

Then T will be the common focus of two hyperbolas. Inverse points of c & c' will be the centres of the two hyperbolas. Inverse points of $a, a,' b, b'$ will be the extremities of the major axes of the two hyperbolas.

These two hyperbolas have parallel asymptotes. Therefore their



major axes and minor axes and all parallel diameters are in the ratio of CT : C′T.

Through C the centre draw any straight line and let it be cut by the two hyperbolas in P, P′, Q, Q.′ Then

$$\frac{CQ\cdot CQ'}{CB\cdot CB'}=\frac{CP^2}{CA^2}$$

But $CB\cdot CB'=CA^2$, as A′B′AB is harmonic range.

$\therefore$ $$CQ\cdot CQ^1=CP^2.$$

Hence any straight line through the centre is cut harmonically by the two hyperbolas.

Therefore in the two hyperbolas: any straight line passing through C or C′ is cut harmonically by the two hyperbolas.

Reciprocating, tangents drawn to the two circles from any point on the chord of contact pcp' or $qc'q'$ form a harmonic pencil.

Question 609 (i).

(M. Bhimasena Rao) :—Show that :

$$\frac{1}{e^{2\pi}-1}+\frac{2^{4n+1}}{e^{4\pi}-1}+\frac{3^{4n+1}}{e^{6\pi}-1}+\ldots=\frac{B_{2n+1}}{4[2n+1]}.$$

Solution by K. B. Madhava.

The series is obviously the integral $\int_0^\infty \frac{x^{4n+1}}{e^{2\pi x}-1}dx.$

But $$B_r=\frac{(2r)\,!}{2^{2r-1}\,.\,\pi^{2r}}\cdot\sum_1^\infty\frac{1}{n^{2r}}.$$

$$\therefore\quad \frac{B_r}{4r}=\frac{\Gamma(2r)}{(2\pi)^{5r}}\cdot\sum_1^\infty\frac{1}{n^{2r}}$$

$$=\int_0^\infty x^{2r-1}\left\{e^{-2\pi x}+e^{-4\pi x}+e^{\quad 6\pi}+\ldots\right\}dx.$$

$$=\int_0^\infty \frac{x^{2r-1}}{e^{2\pi x}-1}\,dx.$$

Hence putting $r=2n+1$,

$$\frac{1}{e^{2\pi}-1}+\frac{2^{4n+1}}{e^{4\pi}-1}+\frac{3^{4n+1}}{e^{6\pi}-1}+\dots=\frac{B_{2n+1}}{4(2n+1)}.$$

Question 619.

(S. Krishnaswami Aiyangar) :—If

$$S_n=\frac{1}{(2n-1)^2(2n-3)^2}+\frac{1}{(2n-5)^2}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n-3)^2}\right\}$$
$$+\frac{1}{(2n-7)^2}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n-3)^2}+\frac{1}{(2n-5)^2}\right\}+\dots\dots,$$

shew that

$$\frac{S_2}{5^2}+\frac{S_3}{7^2}+\frac{S_4}{9^2}+\dots=\frac{\pi^6}{2^6\underline{|6}}$$

Solution by T. P. Trivedi, M.A., LL.B.

Writing down a few terms of the series, we get

$$\frac{1}{1^2\cdot3^2\cdot5^2}+\frac{1}{7^2}\left\{\frac{1}{5^2}\cdot\frac{1}{3^2}+\frac{1}{1^2}\left(\frac{1}{5^2}+\frac{1}{3^2}\right)\right\}$$
$$+\frac{1}{9^2}\left\{\frac{1}{7^2}\cdot\frac{1}{5^2}+\frac{1}{3^2}\left(\frac{1}{5^2}+\frac{1}{7^2}\right)+\frac{1}{1^2}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\right)\right\}+\text{etc},$$

which is clearly equivalent to the product three at a time of the terms of the series $\frac{1}{1^2},\frac{1}{3^2},\frac{1}{5^2},\frac{1}{7^2}$ etc., to infinity.

Now

$$\cos\theta=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\dots;$$

Hence the series multiplied by $\frac{2^6}{\pi^6}$ is equal to the coefficient of $-\theta^6$ in the expansion of $\cos\theta$, and the series is thus equal to $\frac{\pi^6}{2^6\cdot6}$

Question 620.

(S. Krishnaswami Aiyangar) :—Find the value of

$$1-\frac{3^6x^2}{2!}+\frac{5^6x^4}{4!}-\frac{7^6x^6}{6!}+\dots$$

Solution by T. P. Trivedi, M.A., LL.B., A. Narasinga Rao, B.A., and N. Sankara Aiyar, M. A.

We have

$$x\cos x = x - \frac{x^3}{\underline{|2}} + \frac{x^5}{\underline{|4}} - \frac{x^7}{\underline{|6}} + \dots$$

The series evidently is $\left(x\dfrac{d}{dx}\right)^5 (x\cos x)$ which when expanded becomes

$$\cos x(15x^4 - 90x^2 + 1) - \sin x(x^5 - 65x^3 + 31x).$$

Remarks by K. B. Madhava.

By applying formula (3) on p. 184 of the Journal for 1913, we find $\phi(n) = (1+n)^5$, $\Delta\phi(0) = 1$, $\Delta^2\phi(0) = 31$, $\Delta^3\phi(0) = 180$, $\Delta^4(0) = 390$, $\Delta^5\phi(0) = 360$, $\Delta^6\phi(0) = 120$, $\Delta^7\phi(0) = \Delta^8\phi(0) = \dots = 0$.

Hence the series is equal to

$$\cos x(15x^4 - 90x^2 + 1) - \sin x(x^5 - 65x^3 + 31x).$$

Question 621.

(R. Vythynathaswamy) :—If C is the chord of an arc λ of a closed curve, shew that $\int(C\,ds)$ taken over the contour is equal to λ times the perimeter of the locus of centroids of arcs of the curve equal to λ.

Solution by A. Narasinga Rao, B.A.

Let AB, A_1B_1 be consecutive positions of the chord so that

$$\text{arc } AB = \text{arc } A_1B_1 = \lambda.$$

$$\therefore\ AA_1 = BB_1 = ds.$$

Let H be the centroid of the arc A_1 B. Join H to the mid-points of AA_1, BB_1 and divide the joining lines in the ratio of $ds : A_1B_1$ at α, β respectively. Then α, β are the centroids of the arcs AB, A_1B_1.

Also, we have ultimately

$$\alpha\beta : \text{chord } AB = ds ; \text{ arc } AB$$

i.e. $$\alpha\beta : C = ds : \lambda.$$

Thus $\int C\,ds = \lambda \int \alpha\beta$

$= \lambda$ times the corresponding perimeter of the locus of α.

Question 622.

(R. Vythynathaswamy) :—A moving quadrilateral has its four corners on a fixed circle. Shew that at any instant the sum of the angular velocities of the four pedal lines is equal to the sum of the angular velocities of the four corners about the centre.

Solution by A. Narasinga Rao, B.A.

Let ABCD be one of the positions of the moving quadrilateral, and let A_1 B_1 tha points in which the $\perp^s$ from A and B on CD meet the circle ABCD. The pedal lines of A, B *w.r.t.* $\triangle$ formed by the other three points are parallel respectively to BA_1 and A_1B.

Hence the angle between these two

$$=\text{angle between } BA_1 \text{ and } A_1 B$$
$$=\text{angle which AB subtends at the centre O.}$$

Let α', α be the angles which the pedal line of A, and radius OA make with a fixed line ; with similar notations for the points B, C,D and their pedal lines, by what has just been proved,

$$\alpha'-\beta'=\alpha-\beta\,;$$

or $$\alpha'-\alpha=\beta'-\beta=\gamma'-\gamma=\delta'-\delta'=c \text{ (say)}.$$

$\therefore$ $$\alpha'+\beta'+{}'\gamma+\delta'=\alpha+\beta+\gamma+\delta+4c.$$

$\therefore$ $$\frac{d}{dt}(\alpha'+\beta'+\gamma'+\delta')=\frac{d}{dt}(\alpha+\beta+\gamma+\delta).$$

Hence the sum of the angular vels. of the pedal lines is equal to the sum of the angular vels. of the corners about the centre.

Question 627.

(K. V. Anantanarayana Sastri, B.A.) ;—Four spheres of radii a, b, c, d, intersect at right angles. Shew that the volume of the tetrahedron formed by their centres is

$$\tfrac{1}{6}\, abcd\, (a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}.$$

Solution (1) *by R. J. Pocock, T. P. Trivedi, M.A., L.L.B., and*
T. P. Krishnaswami.

If A, B, C, D are the centres of the spheres $AB=\sqrt{a^2+b^2}$, &c.,

$$\cos CAB=\frac{a^2}{\sqrt{a^2+b^2}\sqrt{a^2+c^2},}\ \&c.\,;$$

also volume of tetrahedron

$$=\tfrac{1}{6}.AB.AC.AD.\left\{1-\Sigma\cos^2 CAB+2\cos CAB.\cos BAD.\cos DAC\right\}^{\frac{1}{2}}$$
$$=\frac{1}{6}\left[(a^2+b^2)(a^2+c^2)(a^2+d^2)\left\{1-\sum\frac{a^4}{(a^2+b^2)(a^2+c^2)}+\frac{2a^6}{(a^2+b^2)(a^2+c^2)(a^2+d^2),}\right\}\right]^{\frac{1}{2}}$$
$$=\tfrac{1}{6}\, abcd(a^{-2}+b^{-2}+c^{-2}+d^{-2})^{\frac{1}{2}}, \text{ after reduction.}$$

Question 631.

(S. Narayana Aiyar, M.A.) :—Shew that if $\phi\,(a, b, c, d)$ denotes

$$1+\frac{x}{1}\frac{c-d}{d}\frac{b+x-1}{a+x-1}+\frac{x(x-1)}{1.2}\frac{(c-d)(c-d-1)}{d(d+1)}\frac{(b+x-1)(b+x-2)}{(a+x-1)(a+x-2)}+\dots,$$

then
$$\frac{\phi(a,\, b,\, c,\, d)}{\phi(c,\, d,\, a\ \ b)}=\frac{\Gamma(x+b)\Gamma(x+c)\Gamma(a)\Gamma(d)}{\Gamma(x+a)\Gamma(x+d)\Gamma(b)\Gamma(c)}.$$

Solution by T. P. Trivedi, M.A., LL.B.

From Question 576, we get

$$e^y=\frac{1+y\,\dfrac{a}{b}+\dfrac{y^2}{1\cdot2}\,\dfrac{a(a+1)}{b(b+1)}+\dots\dots}{1+y\,\dfrac{a-b}{b}+\dfrac{y^2}{1.2}\,\dfrac{(a-b)(a-b-1)}{b(b+1)}+\dots}$$

and also

$$=\frac{1+y\,\dfrac{c}{d}+\dfrac{y^2}{1\cdot2}\,\dfrac{c(c+1)}{d(d+1)}+\dots}{1+y\dfrac{c-d}{d}+\dfrac{y^2}{1.2}\,\dfrac{(c-d)(c-d-1)}{d(d+1)}+\dots}$$

Eliminating e^y, we get

$$\left\{1+y\frac{c-d}{d}+\frac{y^2}{1.2}\frac{(c-d)(c-d-1)}{d(d+1)}+\dots\right\}\left\{1+y\frac{a}{b}+\frac{y^2}{1.2}\frac{a(a+1)}{b(b+1)}+\dots\dots\right\}$$
$$=\left\{1+y\,\frac{a-b}{b}+\frac{y^2}{1.2}\,\frac{(a-b)(a-b-1)}{b(b+1)}+\dots\right\}$$
$$\left\{1+y\,\frac{c}{d}+\frac{y^2}{1.2}\,\frac{c(c+1)}{d(d+1)}+\dots\dots\right\}.$$

Equating the coefficient of y^x on both sides, we get

$$\frac{1}{\underline{|x}}\,\frac{a(a+1)\dots(a+x-1)}{b(b+1)\dots(b+x-1)}+\frac{c-d}{d}\cdot\frac{1}{\underline{|x-1}}\,\frac{a(a+1)\dots(a+x-2)}{b(b+1)\dots(b+x-2)}$$
$$+\frac{(c-d)(c-d-1)}{d(d+1)}\,\frac{1}{\underline{|x-2}}\cdot\frac{1}{1.2}\,\frac{a(a+1)\dots(a+x-3)}{b(b+1)\dots(b+x-3)}+\dots$$
$$=\frac{1}{\underline{|x}}\frac{c(c+1)\dots(c+x-1)}{d(d+1)\dots(d+x-1)}+\frac{a-b}{b}\cdot\frac{1}{\underline{|x-1}}\frac{c(c+1)\dots(c+x-2)}{d(d+1)\dots(d+x-2)}$$
$$+-+\dots;$$

whence it is evident that

$$\frac{1}{\underline{|x}}\,\frac{a(a+1)\dots(a+x-1)}{b(b+1)\dots(b+x-1)}\ \phi\,(a,b,c,d)$$
$$=\frac{1}{\underline{|x}}\frac{c(c+1)\dots(c+x-1)}{d(d+1)\dots(d+x-1)}\ \phi\,(c,d,a,b).$$

Thus

$$\frac{\phi(a,b,c,d)}{\phi(c,d,a,b)}=\frac{b(b+1)\ldots(b+x-1)\,c(c+1)\ldots(c+x-1)}{a(a+1)\ldots(a+x-1)d(d+1)\ldots(d+x-1)},$$
$$=\frac{\Gamma(x+b)\Gamma(x+c)\Gamma(a)\Gamma(d)}{\Gamma(x+a)\Gamma(x+d)\Gamma(b)\Gamma(c)}$$

Question 632.

(R. N. Apte, M. A., F. R. A. S.) :—If r_1, $r_2 \ldots r_n$ be the distances of a point P in the plane of a regular polygon from the vertices, find $\Sigma(r_1^2 r_2^2 r_3^2)$ in terms of n, a the radius of the circum-circle of the polygon and c the distance of P from the centre of the circle, the summation extending over all possible groups of vertices.

Solution by T. P. Trivedi, M.A., LL.B., R. Vythynathaswamy, and N. Sankara Aiyar, M.A.

If $A_1A_2A_3\ldots A_n$ be the polygon, O the centre of the circum-circle and the angle POA_1 be θ, supposing P to lie between A_1 and A_n, the angle $POA_2=\theta+\frac{2\pi}{n}$ and so on. It will be at once seen from a figure that

$$PA_1^2=OP^2+OA_1^2-2OP\ OA_1\cos\theta$$

or

$$r_1^2=c^2+a^2-2ac\cos\theta.$$

Similarly

$$r_2^2=c^2+a^2-2ac\cos\left(\theta+\frac{2\pi}{n}\right),\text{ etc.}$$

We are thus to sum up terms of the type

$$\left\{c^2+a^2-2ac\cos\theta\right\}\left\{c^2+a^2-2ac\cos\left(\theta+\frac{2\pi}{n}\right)\right\}$$
$$\times\left\{c^2+a^2-2ac\cos\left(\theta+\frac{4\pi}{n}\right)\right\}$$

that is,

$$(c^2+a^2)^3-2ac(a^2+c^2)^2\left\{\cos\theta+\cos\left(\theta+\frac{2\pi}{n}\right)+\cos\left(\theta+\frac{4\pi}{n}\right)\right\}$$
$$+4a^2c^2(a^2+c^2)\left\{\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)+\ldots+\ldots\right\}$$
$$-8a^3c^3\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right).$$

As the number of such terms is ${}_nC_3$, it will be found that the co-efficient of $-2ac(a^2+c^2)^2$ is

$$\frac{3\,{}_nC_3}{n}\Sigma\cos\theta,\text{ or }\frac{(n-1)(n-2)}{2}\Sigma\cos\theta;$$

the co-efficient of $4a^2c^2(a^2+c^2)$ is $\frac{3_nC_3}{{}_nC_2}\Sigma \cos\theta \cos\left(\theta+\frac{2\pi}{n}\right)$,

or $$(n-2)\,\Sigma \cos\theta\cos\left(\theta+\frac{2\pi}{n}\right);$$

and the co-efficient of $-8a^3c^3$ is

$$ {}_nC_3\,\Sigma \cos\theta \cos\left(\theta+\frac{2\pi}{n}\right) \cos\left(\theta+\frac{4\pi}{n}\right).$$

Moreover, from the expansion of $\cos n\theta$, it is seen that $\cos\theta$, $\cos\left(\theta+\frac{2\pi}{n}\right)$, etc., are the roots of the equation

$$(2\cos\theta)^n-\frac{n}{1}(2\cos\theta)^{n-2}+\frac{n(n-3)}{1\cdot 2}(2\cos\theta)^{n-4}-\ldots 2\cos n\theta=0.$$

Hence $\Sigma\cos\theta=0$; $\Sigma\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)=-\frac{n}{4}$;

and $\Sigma\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right)=0$; so that the summation is

$$\tfrac{1}{6}n(n-1)(n-2)(a^2+c^2)^3-n(n-2)a^2c^2(a^2+c^2)$$

Question 633.

(K. Appukuttan Erady, M.A.):—A particle is projected from a point on the boundary of a circular pond of radius a with the velocity of projection $\sqrt{(2gc)}$. Prove that if all directions of projection are equally probable, the chance of its falling into the pond is

$$\tfrac{1}{2}-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{\left(1-\frac{a}{c}\cos\phi\right)}\,d\phi.$$

Solution by T. P. Trivedi and K. J. Sanjana.

With the usual axes the equation of the circular boundary may be written $r=2a\cos\phi$, ϕ lying between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. If the angle of projection is θ the range of the particle is $2c\sin 2\theta$, and for the favourable cases we must have $c\sin 2\theta \not> a\cos\phi$. Putting $\sin\psi$ for $a\cos\phi\div c$, this requires

$$\sin 2\theta \not> \sin\psi$$

i.e., $$2\theta<\psi \text{ and } >0, \text{ or } >\pi-\psi \text{ and } <\pi.$$

Hence the chance is seen to be

$$\int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{1}{2}\psi}\cos\theta d\theta+\int_{\frac{1}{2}(\pi-\psi)}^{\frac{1}{2}\pi}\cos\theta d\theta\right]d\phi\div 2\pi$$

or $$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} (\sin \tfrac{1}{2}\psi + 1 - \cos \tfrac{1}{2}\psi)\, d\phi.$$

As ψ is evidently positive and $<\frac{1}{2}\pi$, we get

$$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\phi - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (1-\sin\psi)^{\frac{1}{2}} d\phi$$

$$= \tfrac{1}{2} - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \left(1 - \frac{a}{c}\cos\phi\right) d\phi.$$

When $a=c$, we get $\frac{1}{2} - \frac{2}{\pi}(\sqrt{2}-1)$, which is the result of Ex. 3. Ch. XIV, of Todhunter's *Int. Calculus*.

Question 634.

(E. H. NEVILLE, B.Sc.) :—On the section of a surface by a plane through the normal ON at a point O is taken a point S at a small distance f from O, and the plane parallel to the tangent plane at O and distant $1/\lambda$ from O meets the normals at O and S to the surface in points L and P. Shew that LP^2 is approximately $(f^2/\lambda^2)\{(k_n-\lambda)^2 + a_g\}$, where k_n and a_g are the normal curvature and geodesic torsion of the surface along OS. Deduce that if S traces a small contour round O, λ can be chosen so that P traces a curve which is almost a circle with centre L only if the contour traced by S approximates either to a circle with centre O or to an ellipse whose axes are the principal tangents to the surface, and that λ in the first case must be the mean curvature of the surface, in the second case may have either of two values.

Solution by the Proposer.

Let the normal at S to the plane section meet the plane through L in Q. Since SQ and SP are both perpendicular to the tangent at S to the plane section, SQ is the projection of SP on the normal plane and the angle QSP is $f\,a_g$; also the angle between SQ and ON is $f\,k_n$. Hence LQ is $\pm f\{(k_n/\lambda)-1\}$ and QP is $f\,(a_g/\lambda)$, and the first result follows at once.

Again, if D is half the sum of the principal curvatures at O and I is half the difference between them, and if h is the angle between the normal plane OSN and the first principal plane,

$$k_n = \mathrm{D} - \mathrm{I}\cos 2h,\quad a_g = \mathrm{I}\sin 2h$$

and therefore

$$(k_n-\lambda)^2 + a_g^2 = \{(\mathrm{D}-\lambda)^2 + \mathrm{I}^2\} - 2(\mathrm{D}-\lambda)\,\mathrm{I}\cos 2h.$$

For LP to be independent of h, we must have either f a constant and λ equal to D, or f^2 of the form $1/(p-q\cos 2h)$; in the first case S describes a curve approximately circular, in the second case the locus of S is approximately a central conic with axes along the principal tangents and λ is determined by the quadratic equation

$$q(\mathrm{D}-\lambda)^2 - 2p\,\mathrm{I}(\mathrm{D}-\lambda) + q\,\mathrm{I}^2 = 0$$

which has real roots if and only if the conic is an ellipse. Then in fact if a, b are the semi-axes of the ellipse and α, β the principal curvatures of the surface the two values of λ are $(a\alpha-b\beta)/(a-b)$ and $(a\alpha+b\beta)/(a+b)$, and the possible positions of L form a pair of points harmonic with respect to the principal centres.

The importance of the example lies in its application to optics. Rays which emanate from a single point form after any number of reflections and refractions a set of normals to a family of parallel surfaces, the wave-fronts. If rays forming a small pencil enter the final medium through a circular aperture, the section of the pencil near the aperture is approximately an ellipse, and if the axes of this ellipse are approximately the principal tangents to the wave surface, a condition often secured by symmetry, there are usually two planes normal to the pencil on which the cross section of the pencil is a circle. The two circles obtainable generally differ in size, and the smaller of them is the best approximation possible without further reflection or refraction to an image of the point from which the rays originated. This circle is known as the circle of least confusion; mathematically it always exists, but physically it may be virtual if the rays are diverging when they enter the final medium.

Question 635.

(S. KRISHNASWAMI AIYANGAR, B.A.):—If σ_1, σ_2, σ_3 represent KA, KB, KC where K is the symmedian point and m_1, m_2, m_3 the medians of a triangle ABC, shew that

1. $\Sigma\{a(b^2+c^2)\sigma_1 m_1\} = \frac{1}{2}abc(a^2+b^2+c^2) + \frac{3abc}{2}\cdot\frac{a^4+b^4+c^4}{a^2+b^2+c^2}.$

2. $\Sigma\left\{a(b^2+c^2)\,\frac{\sigma_1}{m_1}\right\} = 4ab^{c}.$

3. $\Sigma\left\{bc\frac{m_1}{\sigma_1}\right\} = \frac{3}{2}(a^2+b^2+c^2).$

Solution by K. B. Madhava, N. Sankara Aiyar and others.

The actual areals of K are $\left(\frac{a^2}{S}, \frac{b^2}{S}, \frac{c^2}{S}\right)$, where $S=a^2+b^2+c^2$.

$\therefore \qquad AK^2=\sigma_1{}^2=\frac{b^2c^2}{S^2}(2S-3a^2)$ from Askwith, § 261.

Also, $AG^2=\frac{1}{9}(2b^2+2c^2-a^2)$ so that $m_1{}^2=\frac{1}{4}(2S-3a^2)$.

$$\therefore \qquad \sigma_1 m_1=\frac{bc}{2S}(2S-3a^2).$$

$$\therefore \quad \Sigma a(b^2+c^2)\sigma_1 m_1=2abc.\,S-\frac{3abc}{S}(a^2b^2+b^2c^2+c^2a^2)$$

$$=\frac{1}{2}abc(a^2+b^2+c^2)+\frac{3abc}{2}\cdot\frac{a^4+b^4+c^4}{a^2+b^2+c^2}.$$

Also $$\frac{\sigma_1}{m_1}=\frac{2bc}{S}.$$

$$\therefore \qquad a(b^2+c^2)\frac{\sigma_1}{m_1}=\frac{2abc}{S}(2a^2+2b^2+2c^2)=4abc.$$

Lastly $$\frac{m_1}{\sigma_1}=\frac{S}{2bc}.$$

$$\therefore \qquad \Sigma ba\frac{m_1}{\sigma_1}=\frac{3}{2}(a^2+b^2+c^2).$$

Question 638.

(S. B. Belekar) :—If $\lambda=e^{i\theta}$, shew that the point $a\frac{(\lambda^2+1)}{2\lambda}, \frac{b(\lambda^2-1)}{2i\lambda}$,

lies on $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If λ_1, λ_2, λ_3 be three points as above, the circle passing through them is

$$x^2+y^2-\frac{(a^2-b^2)}{4aN}\{N^2+M+(LN+1)\}x$$

$$-\frac{(a^2-b^2)}{4biN}\{N^2+M-(LN+1)\}y$$

$$+\frac{(MN+L)(a^2-b^2)-2(a^2+b^2)N}{4N}=0,$$

where $L=\Sigma\lambda_1$, $M=\Sigma\lambda_1\lambda_2$, $N=\lambda_1\lambda_3\lambda_3$.

Solution by R. Srinivasan, M.A., T. P. Trivedi and K. J. Sanjana.

As $\lambda = \cos\alpha + i\sin\alpha$, $\lambda^{-1} = \cos\alpha - i\sin\alpha$, $\cos\alpha = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)$, and $\sin\alpha = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right)$. Thus the point mentioned is $a\cos\alpha$, $b\sin\alpha$, which is of course on the given ellipse.

The equation of the circle through the three points $\alpha_1, \alpha_3, \alpha_3$ on this ellipse is given by Salmon in the form

$$x^2+y^2-\frac{2(a^2-b^2)}{a}\cos\frac{\alpha_2+\alpha_3}{2}\cos\frac{\alpha_3+\alpha_1}{2}\cos\frac{\alpha_1+\alpha_2}{2}.x$$

$$-\frac{2(b^2-a^2)}{b}\sin\frac{\alpha_2+\alpha_3}{2}\sin\frac{\alpha_3+\alpha_1}{2}\sin\frac{\alpha_1+\alpha_2}{2}.y$$

$$-\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\left\{\cos(\alpha_1+\alpha_3)+\cos(\alpha_2+\alpha_3)\cos(\alpha_3+\alpha_1)\right\}=0.$$

Now $\Pi\cos\frac{\alpha_2+\alpha_3}{2} = \frac{1}{4}\left\{\Sigma\cos\alpha_1 + \cos(\Sigma\alpha_1)\right\}$

$$=\tfrac{1}{4}\left\{\tfrac{1}{2}\Sigma\left(\lambda_1+\frac{1}{\lambda^1}\right)+\tfrac{1}{2}\left(\lambda_1\lambda_2\lambda_3+\frac{1}{\lambda_1\lambda_2\lambda_3}\right)\right\}$$

$$=\tfrac{1}{8}\left\{L+\frac{M}{N}+N+\frac{1}{N}\right\}=\frac{1}{8N}(N^2+M+LN+1);$$

and $\Pi\sin\frac{\alpha_2+\alpha_3}{2} = \frac{1}{4}\left\{\Sigma\sin\alpha_1 - \sin(\Sigma\alpha_1)\right\}$

$$=\frac{1}{8i}\left\{L-\frac{M}{N}-N+\frac{1}{N}\right\}$$

$$=\frac{1}{8Ni}\left\{LN+1-(N^2+M)\right\},$$

also $\Sigma\cos(\alpha_1+\alpha_2) = \frac{1}{2}\Sigma\left((\lambda_1\lambda_2+\frac{1}{\lambda_1\lambda_2}\right)$

$$=\tfrac{1}{2}\left\{\Sigma(\lambda_1\lambda_2)+(\Sigma\lambda_1)\div\lambda_1\lambda_2\lambda_3\right\}=\frac{MN+L}{2N}.$$

Thus the required equation is

$$x^2+y^2-\frac{a^2-b^2}{4aN}\left\{LN+1+(N^2+M)\right\}$$

$$-\frac{b^2-a^2}{4biN}\left\{LN+1-(N^2+M)\right\}$$

$$+\frac{1}{4N}\left\{(a^2-b^2)(MN+L)-2(a^2+b^2)N\right\}=0.$$

Question 640.

(SELECTED) :—*The Magic Number Cards.*

"This trick consists of six cards with certain of the numbers from 1 to 63 printed on them. The magician asks you to think of a number between 1 and 63 and then presents these cards one at a time with the request that you indicate whether your number is on the card. This done he at once declares the number which you thought of."

Explain the trick and point out any easy extensions of it.

Solution by K. J. Sanjana and V. Anantharaman.

The result follows at once from the fact that in the scale of 2 there are only two digits, 1 and 0, so that any number is the sum of a certain number of powers of 2 (including $1 \equiv 2^0$). With six cards standing for $2^0, 2^1, 2^2, 2^2, 2^4, 2^5$, any number up to the sum of all these (*i.e.* $63 \equiv 111111$ in the scale of 2) may be presented; if in any number some of these powers are not required, the cards corresponding to those powers will not be used. Thus

$23 (= 1 + 2 + 2^2 + 2^4)$ requires the first, second, third and fifth;

$24 (= 2^3 + 2^4)$ requires the fourth and fifth;

25 requires the first, fourth and fifth; and so on.

It is evident that n cards may be used giving numbers up to $2^n - 1$ inclusive. We can also use the scale of 3, which will however require some powers to be subtracted; otherwise the digit 2 may be shown by repeating the number for the power which has this coefficient. Thus

$23 (= 2 \cdot 3^2 + 3 + 2 \cdot 3^0)$ requires two entries on the first, one on the second and two on the third;

$24 (= 2 \cdot 3^2 + 2 \cdot 3)$ requires two on the second and two on the third;

$25 (= 2 \cdot 3^2 \div 2 \cdot 3 + 3^0)$, two on the third, two on the second and one on the first; and so on.

In books on Algebra these properties are generally enunciated thus: "A weight of N lbs. can be balanced by a number of weights of the series 1 lb., 2 lbs., 2^2 lbs......., not more than one weight of each kind being used; and by a number of weights of the series 1 lb, 3 lbs., 3^2 lbs.,......, not more than one weight, each kind being used but *in either scalepan* as may be necessary."

Question 642.

(S. Ramanujan):—Shew that

$$(1)\ \sum_{0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+-\frac{1}{2n+1}\right)5^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{4\sqrt{5}}-\frac{\sqrt{5}}{24}\log\,(2+\sqrt{5})^2.$$

$$(2)\ \sum_{0}^{\infty}\left(1+\frac{1}{3}+\dots+\frac{1}{2n+1}\right)9^{-n}(2n+1)^{-1}$$

$$=\frac{\pi^2}{8}-\frac{3}{8}(\log 2)^2$$

Solution and Remarks by M. Bhimasena Rao.

Let $f(x)\equiv\sum_{n=0}^{\infty}\left(1+\tfrac{1}{3}+\tfrac{1}{5}+\dots\frac{1}{2n+1}\right)\frac{x^{2n+1}}{2n+1}.$

$$=\int_0^x \frac{1}{2}\log\frac{1+x}{1-x}.\ \frac{dx}{(1-x^2)x}$$

$$=\int_0^x \frac{1}{4x}\log\frac{1+x}{1-x}\,d\log\frac{1+x}{1-x}$$

$$=\int_1^y \frac{1}{4}.\ \frac{1+y}{1-y}\log\frac{1}{y}\,d\log\frac{1}{y},\ \text{where } y=\frac{1-x}{1+x}$$

$$=\int_1^y \frac{1}{4}\log y\,\frac{dy}{y}\left(1+\frac{2y}{1-y}\right)$$

$$=\frac{(\log y)^2}{8}+\frac{1}{2}\int_1^y \frac{\log y}{1-y}.\ dy$$

$$=\frac{(\log y)^2}{8}-\frac{1}{2}\log y.\ \log\,(1-y)+\frac{1}{2}\int_1^y \frac{\log\,(1-y)}{y}dy$$

$$=\frac{(\log y)^2}{8}-\tfrac{1}{2}\log y\log\,(1-y)-\frac{1}{2}\int\sum_{1}^{y}\frac{y^n}{x^2}$$

$$=\frac{1}{8}\log y\log\frac{y}{(1-y)^4}-\tfrac{1}{2}\phi(y)+\frac{\pi^2}{12}\qquad\dots\qquad\dots\qquad(1)$$

where $\phi(y)$ stands for $\sum\frac{y^n}{n^2}$.

It may be shown that

(i) $\phi(\tfrac{1}{2})=\frac{\pi^2}{12}-\tfrac{1}{2}(\log 2)^2$

(ii) $\phi\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{10}-\left(\log\frac{\sqrt{5}-1}{2}\right)^2$

(iii) $\phi\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{15}-\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$

[Bromwich : *Infinite Series*, p. 487, Ex. 21.]

For these values of y, the corresponding values of x are $\frac{1}{3}$, $\frac{1}{\sqrt{5}+2}$, $\frac{1}{\sqrt{5}}$, since $x=\frac{1-y}{1+y}$.

$$\therefore\quad f\left(\frac{1}{3}\right)=-\frac{3}{8}(\log 2)^2+\frac{1}{4}(\log 2)^2-\frac{\pi^2}{24}+\frac{\pi^2}{12}=\frac{\pi^2}{24}-\frac{(\log 2)^2}{8}\quad\ldots\ (2)$$

$$f\left(\frac{1}{\sqrt{5}+2}\right)=\frac{1}{8}\log\frac{\sqrt{5}-1}{2}\log\frac{\frac{\sqrt{5}-1}{2}}{\left(1\ \frac{\sqrt{5}-1}{2}\right)^4}$$

$$-\frac{\pi^2}{20}+\frac{1}{2}\left(\log\frac{\sqrt{5}-1}{2}\right)^2+\frac{\pi^2}{12}$$

$$=\frac{\pi^2}{30}-\frac{1}{24}\left\{\log(2+\sqrt{5})\right\}^2\quad\ldots\quad\ldots\ (3)$$

since $\left(\frac{\sqrt{5}-1}{2}\right)^2=\frac{3-\sqrt{5}}{2}, \left(\frac{\sqrt{5}-1}{2}\right)^3=\sqrt{5}-2=(2+\sqrt{5})^{-1}.$

$$f\left(\frac{1}{\sqrt{5}}\right)=\frac{1}{8}\log\frac{3-\sqrt{5}}{2}\log\frac{\frac{3-\sqrt{5}}{2}}{\left(\frac{\sqrt{5}-1}{2}\right)^4}$$

$$-\frac{\pi^2}{30}+\frac{1}{2}\left(\log\frac{\sqrt{5}-1}{2}\right)^2+\frac{\pi^2}{12}$$

$$=\frac{1}{8}\log\left(\frac{\sqrt{5}-1}{2}\right)^2\log\left(\frac{\sqrt{5}-1}{2}\right)^{-2}$$

$$+\frac{1}{2}\left(\log\frac{\sqrt{5}-1}{2}\right)^2+\frac{\pi^2}{20}=\frac{\pi^2}{20}\quad\ldots\ (4)$$

Multiplying (2) by 3 we get the second result, but multiplying (4) by $\sqrt{5}$ we see that the right hand side of the first result should be $\frac{\pi^2}{4\sqrt{5}}$ only. This may be verified, both sides being equal to 1·1034553....

The result corresponding to (3) is

$$\sum_{n=0}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{2n+1}\right)(9+4\sqrt{5})^{-n}(2n+1)^{-1}$$

$$=\pi^2\frac{\sqrt{5}+2}{30}-\frac{\sqrt{5}+2}{24}\left\{\log(2+\sqrt{5})\right\}^2.$$

Question 643.

(K. V. ANANTHANARAYANA SASTRY, B.A.)—Prove that

$$\frac{1}{2n+2}+\frac{1}{2}\cdot\frac{1}{2n+4}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{2n+6}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{2n+8}+\dots=\frac{2\cdot4\cdot6\dots2n}{3\cdot5\cdot7\dots(2n+1)}$$

Solution (1) *by K. B. Madhava and K. J. Sanjana, M. A.*;

(2) *by R. Srinivasan, M.A and T. P. Trivedi, M.A., LL.B.*

(1) We can verify by term-by-term integration that

$$f(x, y)=\int_0^1 t^{x-1}(1-t)^y dt$$

$$=\frac{1}{x}-y.\frac{1}{x+1}+\frac{y(y-1)}{1\cdot2}\cdot\frac{1}{x+2}-\dots$$

and that the latter converges if $y+1$ is positive (see Bromwich, p. 165, Ex. 31); so that putting for the present purpose $y=-\frac{1}{2}$ $x=n+1$, we have

$$\frac{1}{n+1}+\frac{1}{2}\cdot\frac{1}{n+2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{n+3}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{n+4}+\dots=\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}$$

$$=2.\frac{2\cdot4\cdot6\dots2n}{3\cdot5\cdot7\dots(2n+1)}$$

and the result is established.

(2) We have $(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}.\ x+\frac{1\cdot3}{2\cdot4}x^2+\dots$

$$\therefore \qquad x^n(1-x)^{-\frac{1}{2}}=x^n+\frac{1}{2}.\ x^{n+1}+\frac{1\cdot3}{2\cdot4}.x^{n+2}+\dots$$

If we integrate both sides, we get

$$\frac{x^{n+1}}{n+1}+\frac{1}{2}.\ \frac{x^{n-2}}{n+2}+\frac{1\cdot3}{2\cdot4}.\ \frac{x^{n+8}}{n+3}+\dots=\int_0^x\frac{x^n}{\sqrt{1-x}}dx.$$

Putting $x=1$, twice the left side expression in the question

$$=\int_0^1\frac{x^n}{\sqrt{1-x}}dx;$$

Put $x=\sin^2\theta$; this integral

$$=\int_0^{\frac{\pi}{2}} 2\sin^{2n+1}\theta d\theta$$

$$=2.\frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(n+\frac{3}{2}\right)}$$

$$=\frac{1\cdot2\cdot3\dots\dots n}{\frac{1}{2}\frac{3}{2}\frac{5}{2}\dots\dots\left(n+\frac{1}{2}\right)}$$

$$=2.\frac{2\cdot4\cdot6\dots\dots2n}{1\cdot3\cdot5\dots\dots(2n+1)}$$

Question 644.

(K. V. ANANTHANARAYANA SASTRY, B.A.) :—Show that the length of the fourth positive pedal of a loop of the Lemniscate of Bernoulli is given by

$$18a\int_0^1 \frac{x^8dx}{\sqrt{(1-x^4)}}.$$

Solution by K. B. Madhava ; T. P. Trivedi, M.A., LL.B.
and R. Srinivasan, M.A.

The fourth positive pedal of the Lemniscate is

$$r^{2/9}=a^{2/9}\cos\tfrac{2}{9}\theta \text{ (Edwards : P. 166, Ex. 1)}$$

so that

$$\frac{ds}{d\theta}=a\cos^{7/2}\frac{2}{9}\theta;$$

and therefore the length of the loop is given by $9a\int_0^{\frac{\pi}{2}}\cos^{\frac{7}{2}}\phi\, d\phi.$

This is easily seen to be equal to

$$18\,a\int_0^1\frac{x^8dx}{\sqrt{1-x^4}},\quad \text{by the substitution } \cos\phi=x^2.$$

Question 646.

(R. SRINIVASAN, M.A.) :—If $xy+yz+zx=1$, show that

$$\Sigma\left\{(1+x^2)^{\frac{1}{2}}-x\right\}\left\{(1+y^2)^{\frac{1}{2}}-y\right\}=1.$$

Solution (1) *by J. C. Swaminarayan, M.A. and K. B. Madhava,*
(2) *by K. J. Sanjana, C. Bhaskaraiya and H. V. Venkataramniengar.*

(1) Here

$$1+x^2=x^2+xy+yz+zx=(x+y)(x+z).$$

$$\begin{aligned}
\therefore\quad &\Sigma\left\{(1+x^2)^{\frac{1}{2}}-x\right\}\left\{(1+y^2)^{\frac{1}{2}}-y\right\}\\
&=\Sigma(1+x^2)^{\frac{1}{2}}(1+y^2)^{\frac{1}{2}}-\Sigma(1+x^2)^{\frac{1}{2}}(x+y)+1.\\
&=\Sigma(x+y)\left\{(x+z)(y+z)\right\}^{\frac{1}{2}}-\Sigma(x+y)(1+z^2)^{\frac{1}{2}}+1\\
&=\Sigma(x+y)(1+z^2)^{\frac{1}{2}}-\Sigma(x+y)(1+z^2)^{\frac{1}{2}}+1\\
&=1
\end{aligned}$$

(2) If $x=\tan\alpha$, $y=\tan\beta$, $z=\tan\gamma$ and $\alpha+\beta+\gamma=\dfrac{\pi}{2}$, then $xy+yz+zx=1$.

In this case $\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\left(\frac{\pi}{4}-\frac{\beta}{2}\right)+\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=\frac{\pi}{2}$

and $$\sum \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)=1 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

But $$\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\frac{1-\cos\left(\frac{\pi}{2}-\alpha\right)}{\sin\left(\frac{\pi}{2}-\alpha\right)}=\sec\alpha-\tan\alpha=(1+x^2)^{\frac{1}{2}}-x;$$

from which the result follows at once.

Question 647.

(K. Appukuttan Erady, M.A.) :—The curves $\alpha=$constant, and $\beta=$constant, where α, β are conjugate functions of (x, y) pass through a point. Prove that the curvatures of the two curves at the point are

$$-\frac{\partial h}{\partial \alpha} \text{ and} -\frac{\partial h}{\partial \beta}$$

where $$h=\left\{\frac{\partial(\alpha, \beta)}{\partial(x, y)}\right\}^{\frac{1}{2}}.$$

Solution by J. C. Swaminarayan, M.A.

We know that $\alpha_x=\beta_y$, $\beta_x=-\alpha_y$,

$$\alpha_{xx}+\alpha_{yy}=0,\ \beta_{yx}+\beta_{yy}=0,\ x\alpha=y\beta \text{ and } x\beta=y_\alpha.$$

By hypothesis, $\alpha+i\beta=f(x+iy)$

$$\therefore \qquad \alpha_x+i\beta_x=f'(x+iy)$$

and $$f'(x+iy)\left\{x_\alpha+iy_\alpha\right\}=1.$$

$$\therefore \qquad (\alpha_x-i\alpha_y)(x_\alpha+iy_\alpha)=1$$

$$\therefore \qquad \alpha_x.\ x_\alpha+\alpha_y{\cdot}y_\alpha=1 \text{ and } \alpha_y.\ x_\alpha-\alpha_x{\cdot}y_\alpha=0.$$

$$\therefore \qquad x_\alpha=\frac{\alpha_x}{\alpha_x^2+\alpha_y^2} \text{ and } y_\alpha=\frac{\alpha_y}{\alpha_x^2+\alpha_y^2}.$$

$$\therefore \quad -\frac{\partial h}{\partial \alpha}=-\frac{\partial}{\partial \alpha}\left\{\alpha_x\beta_y-\alpha_y\beta_x\right\}=-\frac{\partial}{\partial \alpha}\left\{\alpha_x^2+\alpha_y^2\right\}$$

$$=\frac{-\left[\alpha_{xx}(\alpha_x)^2+2\alpha_x\alpha_y\alpha_{xy}+\alpha_{yy}(\alpha_y)^2\right]}{(\alpha_x^2+\alpha_y^2)^{\frac{3}{2}}}$$

$$=\frac{\alpha_{xx}\alpha_y^2-2\alpha_x\alpha_z\alpha_{xy}+\alpha_{yy}\alpha_y^2}{(\alpha_x^2+\alpha_y^2)^{\frac{3}{2}}}$$

$=$curvature at the point of intersection on the curve $\alpha=$constant.

Question 648.

(K. Appukuttan Erady, M.A.) :—Tangents TP, TQ are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the parallelogram TP SQ is completed. Prove that the area of the locus of S, as T completes a circuit along $\frac{x^2}{\lambda^2}+\frac{y^2}{\mu^2}=1$, is

$$\pi\lambda\mu+2\pi ab\left(\frac{a^2}{\lambda^2}+\frac{b^2}{\mu^2}-2\right).$$

Solution by J. C. Swaminarayan, M.A.

If TS is produced, it will pass through the centre C of the ellipse. If $\alpha+\beta$, $\alpha-\beta$ are the eccentric angles of P, Q

$$\frac{a\cos\alpha}{\cos\beta}=\lambda\cos\phi,\quad \frac{b\sin\alpha}{\cos\beta}=\mu\sin\phi,$$

since T lies on the ellipse $\frac{x^2}{\lambda^2}+\frac{y^2}{\mu^2}=1$.

Hence $\sec^2\beta=\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi$.

The coordinates of S are $\left(\frac{a\cos\alpha}{\cos\beta}\cos 2\beta,\ \frac{b\sin\alpha}{\cos\beta}\cos 2\beta\right)$

Thus the polar co-ordinates of S will be

$$\left[\left\{\lambda^2\cos^2\phi+\mu^2\sin^2\phi\right)^{\frac{1}{2}}\left(\frac{2}{\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi}-1\right),\ \tan^{-1}\left(\frac{\mu}{\lambda}\tan\phi\right)\right\}\right]$$

Hence the area traced out by S in a complete circuit

$$=\int_0^{\frac{\pi}{2}} r^2 d\theta=2\lambda\mu\int_0^{\frac{\pi}{2}}\left\{\frac{2}{\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi}-2\right\}^2 d\phi$$

$$=2\lambda\mu\int_0^{\frac{\pi}{2}}\left\{\frac{4}{\left(\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi\right)^2}-\frac{4}{\left(\frac{\lambda^2}{a^2}\cos^2\phi+\frac{\mu^2}{b^2}\sin^2\phi\right)}+1\right\}d\phi$$

$$=2\lambda\mu\left[4.\frac{\pi a^3b^3}{\lambda^3\mu^3}\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}\right)-4.\frac{\pi ab}{2\lambda\mu}+\frac{\pi}{2}\right]$$

$$=\pi\lambda\mu+2\pi ab\left(\frac{a^2}{\lambda^2}+\frac{b^2}{\pi^2}-2\right)$$

Additional Solutions by A. Narasinga Rao and T. P. Krishnanswami.

QUESTIONS FOR SOLUTION.

706. (N. Sankara Aiyar, M.A.):—If n is any number, show that

$$\frac{1}{(n+1)(n+2)}+\frac{1}{2}\,\frac{1}{(n+2)(n+3)}+\frac{1\cdot3}{2\cdot4}\,\frac{1}{(n+3)(n+4)}+\ldots$$
$$=2^{2n+2}\frac{\Gamma(n+1)\ \Gamma(n+2)}{\Gamma\ \ (2n+4)}$$

707. (D. Krishnamurti):—Evaluate the integral

$$\int_{0}^{a} y\,dy\int_{0}^{y}(a^{3}-x^{3})^{\frac{1}{3}}dx.$$

708. (A. Narasinga Rao):—Find the maximum and minimum values of $x+2y^{2}+3z^{3}+4w^{4}$, when $4x+3y^{2}+2z^{3}+w^{4}=62$.

709. (R. Srinivasan, M.A.):—The common tangent to the nine points circle of a triangle ABC and the ex-circle opposite to A meets BC in D; E and F are corresponding points for CA and AB. Show that AD, BE, CF are concurrent.

710. (R. Srinivasan, M.A):—Investigate the motion of a rough circular hoop which rolls and describes a circle on a horizontal table keeping its plane inclined at a constant angle to the table.

711. (R. Vythynathaswami):—A pack of mn cards is dealt round face upwards in a circle so as to constitute m heaps of n cards each. The heaps are then placed one upon another in order beginning with the rth. heap; and the cards are again dealt round starting as before. If the same operation is repeated after each dealing, investigate the number of shuffles necessary to bring the cards back to the original positions. Also, if $n>m$, find, after how many arrangements, each heap consists of cards which were in different heaps initially.

712. (J. C. Swaminarayan, M.A.):—If $a>b$ and

$f(a,b)=\int_{0}^{\pi}\log\,(a+b\cos\theta)d\theta$, prove that

$$f(a,\,b)=\tfrac{1}{2}\,f\left(a^{2}-\frac{b^{2}}{2},\ \frac{b^{2}}{2}\right)=\pi\log\left(\frac{a+\sqrt{a^{2}-b^{2}}}{2}\right).$$

713. (*Communicated by* K. V. Anantanarayana Sastri, B.A.):—Find the value of the infinite continued fraction

$$\frac{1}{1+}\,\frac{1}{3+}\,\frac{16}{5+}\,\frac{81}{7+}\cdots\frac{n^{4}}{2n+1}+\ldots\ldots$$

714. (K. B. Madhava) :—Find the co-efficient of x^n in

$$\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+\ldots\ldots$$

715. (Martyn M. Thomas, B.A.) :—Prove that

$$\int_0^{\frac{\pi}{2}} (\sin x \log \sin x)(\cos x \log \cos x)dx=\frac{12-\pi^2}{48}.$$

716. (S. Krishnaswami Aiyangar, B.A.) :—Shew how to find the sum of

$$1+\frac{a}{1\,!}+\frac{2^r a^2}{2\,!}+\frac{3^r a^3}{3\,!}+\ldots\frac{p^r a^p}{p\,!}+\ldots$$

717. (C. Krishnamachary) :—Shew that

$$\begin{aligned}\phi(x+h)-\phi(x+3h)+\phi(x+5h)-\ldots\\ =\phi(x-h)-\phi(x-3h)+\phi(x-5h)-\ldots\end{aligned}$$

718. (S. Narayana Aiyar, M.A.) :—If

$$A_n=\frac{(a-b)(a-bx)(a-bx^2)\ldots\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^n)}$$

and

$$B_n=\frac{(b-a)(b-ax)(b-ax^2)\ldots\ldots\ldots(b-ax^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^n)},$$

then show that

(1) $A_n+A_{n-1}\cdot B_1+A_{n-2}\cdot B_2+\ldots\ldots+A_2B_{n-2}+A_1B_{n-1}+B_n=0.$

(2) $A_1\cdot B_n+2A_2\cdot B_{n-1}+3A_3\cdot B_{n-2}+4A_4\cdot B_{n-3}+\ldots\ldots(n+1)\cdot A_{n+1}$

$$=\frac{a^{n+1}-b^{n+1}}{1-x^{n+1}}.$$

719. (J. C. Swaminarayan, M.A.) :—Determine the limit, when $n\to\infty$, of

$$\frac{e^n\{1\cdot3\cdot5\ldots\ldots(2n-1)\}}{(2n+1)^n}.$$

720. (A. C. L. Wilknson, M.A., F.R.A.S.) :—In a paper in the J. I. M. S. Vol. IV, p. 162, on "Approximations in Curve Tracing", Mr. M. T. Naraniengar discusses two examples of cubic asymptotes. Prove the following results ;

(i) For the curve $x^2(y-x)^3=ay^4$, the family of cubics

$$(y-x)^3=36^2-8xy+xy^2-13x+17y+c$$

has contact of the 7th order at infinity, the cubic having contact of the 8th order consisting of the line at infinity repeated three times.

(ii) For the curve $x(y-x)^3=ay^2(3x-y)$, the cubic

$$(y-x)^3=3ax^2y-ax^2-3ay+a^2x+2a^3$$

has contact of the 9th order at infinity.

721. (A. C. L. Wilkinson, M.A., F. R. A. S.) :—A straight line AB of constant length slides between two rectangular axes, and circles are drawn touching the axes and the line AB; show that the locus of the points of contact of these circles with AB consists of four trinodal quartics whose equations referred to the triangle formed by the nodes is

$$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2-4\alpha\beta\gamma(\alpha+\beta+\gamma)=0.$$

722. (S. Ramanujan) :—Solve completely $x^2=a+y$; $y^2=a+z$; $z^2=a+u$; $u^2=a+x$; and deduce that if

$$x=\sqrt{5+\sqrt{5+\sqrt{5-\sqrt{5+x}}}},$$

then $$x=\tfrac{1}{2}(2+\sqrt{5}+\sqrt{15-6\sqrt{5}});$$

and if $$x=\sqrt{5+\sqrt{5-\sqrt{5-\sqrt{5+x}}}},$$

then $$x=\tfrac{1}{4}\left(\sqrt{5}-2+\sqrt{13-4\sqrt{5}}+\sqrt{50+12\sqrt{5}-2\sqrt{65-20\sqrt{5}}}\right).$$

723. (S. Ramanujan) :—If $[x]$ denotes the greatest integer in x and n is any positive integer show that

(i) $\left[\frac{n}{3}\right]+\left[\frac{n+2}{6}\right]+\left[\frac{n+4}{6}\right]=\left[\frac{n}{2}\right]+\left[\frac{n+3}{6}\right]$

(ii) $\left[\frac{1}{2}+\sqrt{n+\tfrac{1}{2}}\right]=\left[\frac{1}{2}+\sqrt{n+\tfrac{1}{4}}\right]$

(iii) $\left[\sqrt{n}+\sqrt{n+1}\right]=\left[\sqrt{4n+2}\right]$

724. (S. Ramanujan) :—Show that

(i) $$\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\tan^{-1}\frac{1}{2n+5}+\ldots\text{to } n \text{ terms}$$
$$=\tan^{-1}\frac{1}{1+2\cdot1^3}+\tan^{-1}\frac{1}{1+2\cdot2^3}+\tan^{-1}\frac{1}{1+2\cdot3^3}+\ldots\text{to } n \text{ terms.}$$

(ii) $$\tan^{-1}\frac{1}{(2n+1)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+3)\sqrt{3}}+\tan^{-1}\frac{1}{(2n+5)\sqrt{3}}+\ldots\text{to } n \text{ terms}$$
$$=\tan^{-1}\frac{1}{(\sqrt{3})^3}+\tan^{-1}\frac{1}{(3\sqrt{3})^3}+\tan^{-1}\frac{1}{(5\sqrt{5})^3}+\ldots\text{to } n \text{ terms.}$$

List of Periodicals Received.

(From 16th September to 15th November 1915.)

1. Annals of Mathematics, Vol. 17, No. 1, September 1915.
2. Astrophysical Journal, Vols. 41 and 42, Nos. 5, 1 and 2, June, July and September 1915.
3. Bulletin of the American Mathematical Society, Vol. 22, No. 1, October 1915.
4. Bulletin des Sciences Mathematiques, Vol. 39, August and September 1915.
5. Educational Times, September, October and November 1915, (6 Copies).
6. L'Intermediaire des Mathematiciens, Vol. 22, Nos. 6, 7, 8 and 9, June, July, August and September 1915.
7. Journal de Mathematiques Elementaires, Vol. 39, No. 19.
8. Messenger of Mathematics, Vol. 45, No. 3, July 1915.
9. Philosophical Magazine, Vol. 30, Nos. 177 and 178, September and October 1915.
10. Popular Astronomy, Vol. 23, No. 8, October 1915, (3 Copies).
11. Proceedings of the Royal Society of London, Vol. 91, No. 633, September 1915.
12. School Science and Mathematics, Vol. 15, Nos. 3 and 7, March and October 1915, (2 Copies).
13. Transactions of the Royal Society of London, Vol. 215, Nos. 533 to 536.
14. American Mathematical Monthly, Vol. 22, No. 7 September 1915.

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